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Nonlinear dynamics of fractional order Duffing system

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ABSTRACT

In this paper, we analyze the nonlinear dynamics of fractional order Duffing system. First, we present the fractional order Duffing system and the numerical algorithm. Second, nonlinear dynamic behaviors of Duffing system with a fixed fractional order is studied by using bifurcation diagrams, phase portraits, Poincare maps and time domain waveforms. The fractional order Duffing system shows some interesting dynamical behaviors. Third, a series of Duffing systems with different fractional orders are analyzed by using bifurcation diagrams. The impacts of fractional orders on the tendency of dynamical motion, the periodic windows in chaos, the bifurcation points and the distance between the first and the last bifurcation points are respectively studied, in which some basic laws are discovered and summarized. This paper reflects that the integer order system and the fractional order one have close relationship and an integer order system is a special case of fractional order ones.

1. Introduction

Fractional calculus was proposed about three hundred years ago. It has attracted many investigators [1–2]. Many famous fractional order systems, such as Rössler system, Lorenz system, Chua's circuit, Duffing system and so on, have been studied [3–9]. In view of the fact that fractional calculus provides another good way to describe, predict and control physical systems accurately, it has been applied to control system, physics and system modeling [10–14].

Duffing system has been applied in many fields, for example, fluid flow induced vibration, large amplitude oscillation of centrifugal governor systems and mathematical modeling and so on [15–17]. Perturbation methods and harmonic balance methods are used to study Duffing system [18,19]. Fractional order Duffing system has been studied recently [20-22]. Some researchers studied fractional damped Duffing systems and found some new dynamic behaviors [23–26]. However, only few fractional orders are analyzed and the basic laws haven't been summarized systematically.

Motivated by the above discussions, there are three novel points in this paper, compared to the prior work. First, some interesting dynamic behaviors as for example that the width of period-three window is narrower than the period-five window occur in the fractional order Duffing system. Second, we innovatively study how the tendency of dynamic motion varies with fractional order, how periodic windows in chaos vary with fractional order, how the location of bifurcation points varies with fractional order and how the distance between the first bifurcation point and the last bifurcation point varies with fractional order. Third, when fractional order varies, the value of excitation frequency where the bifurcation occurs has minimum and the distance between the first bifurcation point and the last bifurcation point has minimum. This is an important discovery.

This paper is organized as follows: Section 2 presents the fractional order Duffing system. In Section 3, the nonlinear dynamic behaviors of fractional order Duffing system are studied in detail and Section 4 concludes this paper.

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2. Fractional order Duffing system

2.1. Definition of fractional derivative

There are three most frequently used for fractional derivative: Grunwald–Letnikov, Riemman–Liouville, and Caputo definitions. The Caputo fractional derivative is used in this paper, which is defined by:

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau, \tag{1} $$

in which $n-1<q<n$. Compared Caputo definition with Grunwald–Letnikov and Riemman–Liouville, the most difference is that the initial conditions of the fractional differential equations with Caputo derivatives take on the same form as those for the integer order ones. It has physical meaning and is very appropriate for practical problems [27]. Therefore, Caputo derivative is applied to this paper.

2.2. Numerical algorithms

To solve a fractional differential equation, two methods, which are the time domain approach and frequency domain approach [28–31], are mainly used. We use the Adams–Bashforth–Moulton predictor–corrector scheme [31], a time domain approach in this paper. For a fractional order system, the time domain method is complicated and consumes a very long simulation time, but it is more accurate [29].

The Adams–Bashforth–Moulton type predictor–corrector scheme is based on the following fractional differential equation:

$$ \begin{cases} D_t^q y(t) = f(y(t), t), \\ y^{(k)}(0) = y_0^k, \quad k = 0, 1, \ldots, m-1 \end{cases}, \tag{2} $$

which is equivalent to the Volterra integral equation:

$$ y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, y(\tau)) d\tau. \tag{3} $$

Discretizing the Volterra equation by setting $t_n = nh$ ($n = 0, 1, \ldots, N$) and $h = T_{sim}/N$, we can obtain

$$ \begin{aligned} y_h(t_{n+1}) = {} & \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^q}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})) \\ & + \frac{h^q}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_n(t_j)), \end{aligned} \tag{4} $$

where

$$ a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} \\ \quad +2(n-j+1), & 1 \le j \le n, \\ 1, & j = n+1. \end{cases} \tag{5} $$

The predictor $y_h^p(t_{n+1})$ is given by

$$ y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^k + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_n(t_j)), \tag{6} $$

in which

$$ b_{j,n+1} = \frac{h^q}{q} ((n+1-j)^q - (n-j)^q). \tag{7} $$

The error estimate is

$$ \max_{i=0,1,\ldots,N} |y(t_i) - y_h(t_i)| = O(h^p), $$

in which $p = \min(2, 1+q)$.

2.3. Fractional order Duffing system

The famous Duffing system is

$$ m\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + kx(t) + ax^3(t) = f \sin(\omega t), \tag{8} $$

where m is the mass, c is the damping factor, k is the coefficient of linear rigidity, a is the coefficient of nonlinear stiffness, f is the amplitude of the excitation and ω is the frequency of the excitation.

Thus, the fractional order version of Duffing system is:

$$ \begin{cases} \dfrac{d^\alpha x}{d^\alpha t} = y \\ \dfrac{d^\beta y}{d^\beta t} = \dfrac{1}{m}(f \sin(\omega t) - ax^3 - kx - cy), \end{cases} \tag{9} $$

where α and β are the fractional orders of the fractional order Duffing system ($0 < \alpha, \beta < 2$). In this paper, we use the Adams–Bashforth–Moulton predictor–corrector scheme to solve the fractional order Duffing equations and get the numerical results.

In the following sections, we fix $c = 0.9$, $f = 0.6$, $m = 1$, $k = -1$, $a = 1$ and let parameter ω vary. Also the initial conditions are $x(0) = 0$, $y(0) = 0$.

3. Nonlinear dynamic analysis of fractional order Duffing system

We will study the nonlinear dynamic behaviors of the above fractional order Duffing system in detail. In Section 3.1, we will investigate the fractional order Duffing system with $\alpha = 1.2$ and $\beta = 0.8$. Moreover, a series of bifurcation diagrams are shown to discover and summarize the basic laws of fractional order Duffing systems in Section 3.2.

3.1. Fractional order Duffing system with $\alpha = 1.2$ and $\beta = 0.8$

In order to analyze the fractional order Duffing system, the bifurcation diagram has been shown. Fig. 1(a) exhibits the bifurcation diagram of x versus ω of fractional order Duffing system on the interval $0.43 < \omega < 1.22$, with fractional order $\alpha = 1.2$ and $\beta = 0.8$.

As ω decreases from $+\infty$ to 0.43, the fractional order Duffing system goes through a cascade of period-doubling bifurcations to chaos, and it returns into single-periodic orbit from chaos eventually.

For $\omega > 1.171$, as shown in Fig. 1(a), the fractional order Duffing system exhibits period-1 motion. The phase portrait and Poincare map are depicted at $\omega = 1.2$ in Fig. 1(b1) and (b2), respectively. Clearly, the phase portrait is a universal limit cycle; correspondingly, the Poincare map exhibits an isolated point. The critical point of period-doubling bifurcation is at $\omega = 1.171$, where a pitchfork bifurcation emerges, resulting in a transition of the phase portrait from a limit cycle to a period-two limit cycle. From Fig. 1(c1) and (c2), the period-two phase portrait and the Poincare map at $\omega = 1.11$