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Homoclinic attractors in discontinuos iterated function systems

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Iterated function systems which consist of discontinuous maps are shown to be able to have attractors in the sense of Hutchinson. Moreover, it is demonstrated that discontinuous systems often admit strict attractors which are non-invariant, so one can call them homoclinic attractors. The existence of such attractors relates to the notion of a fast basin.

1. Introduction

Over the years, an interest in non-contractive IFSs has grown, e.g., [1,5,10,16,17,19,20]. Various kinds of attractors for possibly discontinuous IFSs were offered using the language of multivalued maps, e.g., [11,13–15]. Still, the research on discontinuous systems is mostly occupied by the dynamics of a single map, e.g., [18] (compare with [1]).

In the present paper we would like to ask about the existence of strict attractors in discontinuous IFSs. We tackle two characteristic cases which concern modifications of a given IFS $\mathcal{F}$ with an attractor A.

(i) Can we modify maps comprising $\mathcal{F}$ so that they become discontinuous and the modified system $\widetilde{\mathcal{F}}$ admits A as an attractor?

(ii) Can we modify the dynamics of $\mathcal{F}$ on A so that the modified system $\widetilde{\mathcal{F}}$ flushes part of A outside A and A is an attractor of $\widetilde{\mathcal{F}}$?

Question (i) is easily addressed. The positive answer to (ii) gives rise to “homoclinic attractors”, i.e., non-invariant compacta attracting nearby sets. Surprisingly, the concept of a fast basin, introduced implicitly in [4] (cf. [3]) by Barnsley and Vince in their quest for fractal generalization of analytic continuation, allows to formulate criteria on the existence of homoclinic attractors in a very particular class of IFSs. The fast basin of an attractor A is the set of those points (in the space containing A) whose orbits fall in A after finite number of iterations; see Section 3 for the precise definition. It is worth noticing that fast basins are related to infinite fractal manifolds, e.g., [21].

The reader should be aware that we deviate from the standard meaning of the adjective “homoclinic”. A homoclinic point is never a homoclinic attractor, nor vice versa.

2. Discontinuous IFSs

By an *iterated function system (IFS)* $\mathcal{F}$ we understand a finite collection of (not necessarily continuous) maps $f_i: X \to X$, $i = 1, \ldots, N$, acting on a metric space (X, d). We write $\mathcal{F} = (X; f_i : i = 1, \ldots, N)$. We tacitly assume that each function f_i maps compact sets onto relatively compact sets.

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The symbol $\mathcal{K}(X)$ stands for the family of nonempty compact subsets of X endowed with the Hausdorff distance d_H, e.g., [8]. The *Hutchinson operator* $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is defined via

$$F(S) = \bigcup_{i=1}^{N} \overline{f_i(S)}\ , \text{for every } \ S \in \mathcal{K}(X).$$

The k-fold composition of F is written as F^k. For systems of continuous maps one usually omits closures in the definition of the Hutchinson operator, but in general closures are necessary, cf. [9,13,14].

The *strict attractor* of $\mathcal{F}$ is a nonempty closed set $A \subset X$ such that

$$F^k(S) \to A, k \to \infty, \text{ with respect to } d_H, \qquad (1)$$

for nonempty compact subsets $S \subset U$ of some open neighbourhood $U \supset A$. An IFS can have more than one strict attractor; it is a local concept.

The maximal open neighbourhood U present in the definition of a strict attractor is called the *basin* of an attractor and denoted by $B(A)$. As it had been proven in [3] for systems of continuous maps, the basin of an attractor is well defined. The justification given therein relies only on the normal separation of closed sets in X and the set-algebraic additivity of the Hutchinson operator F, so no continuity of F is needed at all.

If the maps f_i are continuous, then $F : (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ is continuous too, cf. [11]. Hence, the attractor is invariant, $F(A) = A$. Even if the maps f_i are not continuous, but the multivalued map $\varphi : X \to \mathcal{K}(X)$, $\varphi(x) := F(\{x\})$ for $x \in X$ is upper semicontinuous, then the attractor must be invariant, cf. [14].

Despite the first impression, a system which comprises discontinuous maps can have a strict attractor, even an attractor satisfying the original Hutchinson definition [9].

Example 1 (Discontinuous IFS with a Hutchinson attractor)**.** Let X be a Banach space, $E \neq X$ its nonempty subset, and $f: X \to X$,

$$f(x) = \begin{cases} x/2, & x \in E \\ 0, & x \notin E. \end{cases}$$

Then $A = \{0\}$ is the Hutchinson attractor of $(X; f)$: $F^k(S) \to A = F(A)$ for every nonempty closed bounded $S \subset X$. Indeed, $d_H(F(S), A) \le \frac{1}{2} \cdot d_H(S, A)$. Although f is contractive at 0: $d(f(x), f(0)) \le \frac{1}{2} \cdot d(x, 0)$, $x \in E$, it is not continuous at points in the boundary of E distinct from 0.

The construction can be generalized as follows. Consider a continuous function $g: X \to X$ which maps bounded sets onto bounded sets. Assume that g composed m times with itself, $g^m = g \circ .. \circ g$, is a contraction with Lipschitz constant $L < 1$. Note that the map g^m admits a unique fixed point x_0 which is also a unique fixed point of g, cf. [7, chap.I Section 1.6 (A.1)]. Maps with contractive iterate are often called eventually contractive. They appear naturally in the theory of integral equations, cf. [12, chap.2 Section 15].

Take any set $E \subset X$ with nonempty boundary so that $x_0 \notin E$. Define, as previously,

$$f(x) = \begin{cases} g(x), & x \in E \\ x_0, & x \notin E. \end{cases}$$

Then $A = \{x_0\}$ is the Hutchinson attractor of $(X; f)$. To see this, one estimates

$$\begin{aligned} & d_H(F^{mk+i+m}(S), A) \\ & \le d_H(\overline{g^{mk+i+m}(S)} \cup A, A) \\ & = d_H(g^{mk+i+m}(S), A) \\ & \le L \cdot d_H(g^{mk+i}(S), A) \le L^{k+1} \cdot d_H(g^i(S), A), \end{aligned}$$

where $S \subset X$ is a nonempty closed bounded set, $i \in \{0, 1, \ldots, m-1\}$, $k \to \infty$.

In fact, we can take any IFS $\mathcal{F} = (X; f_i : i = 1, \ldots, N)$ consisting of continuous maps, which admits a strict attractor A, and modify maps f_i to $\widetilde{f_i} : X \to X$ so that $\widetilde{f_i}(x) \in A \cup \{f_i(x)\}$ for $x \in X \setminus A$, $\widetilde{f_i}|_A = f_i|_A$, to obtain a similar effect to the one exhibited in the example above. Nonetheless, the attractor A of the modified system $\widetilde{\mathcal{F}} = (X; \widetilde{f_i} : i = 1, \ldots, N)$ would be still invariant as it was for the original system $\mathcal{F}$.

In the next section, we explore the possibility that the attractor can be a non-invariant set.

Before we move forward, let us comment on a hidden contractivity of all maps in Example 1. The key is the converse of the Banach theorem which roughly states that any map for which the scheme of successive iterations converges to a unique fixed point, actually is a contraction under suitable metric, cf. [7, chap.I Section 1.7 p.24]. Therefore, eventually contractive maps are contractions, though after changing a reference metric. Similarly, if the IFS admits a strict attractor (respectively Hutchinson attractor), then the Hutchinson operator is a contraction on the hyperspace of nonempty compact (respectively nonempty closed bounded) sets under some metric different from the Hausdorff distance. Thus, in a sense, we cannot escape from contractivity. What may be surprising to the reader is the fact that still there exist IFSs which admit attractor although all the maps comprising IFS are non-contractive under any complete metric! See for instance [2, Example 2.1].

3. Homoclinic attractors

Let $\mathcal{F} = (X; f_i, i = 1, \ldots, N)$ be an IFS with a strict attractor A and its basin of attraction $B(A) \neq A$. Let us select any $b \in B(A) \setminus A$. We put $\widetilde{f_i}(a) := b$ for $a \in A$ and $\widetilde{f_i}|_{X \setminus A} = f_i|_{X \setminus A}$. Thus, we have a (generically) discontinuous IFS $\widetilde{\mathcal{F}_b} = (X; \widetilde{f_i}, i = 1, \ldots, N)$. We write $\widetilde{F}$ for the Hutchinson operator associated with any modification $\widetilde{\mathcal{F}}$ of the original IFS $\mathcal{F}$.

We investigate when A is a strict attractor of $\widetilde{\mathcal{F}_b}$. Since $\widetilde{F}(A) \not\subset A$, in such a case A would be called a *homoclinic attractor*. Of course this phenomenon can occur for discontinuous systems only.

It turns out that the properties of the modified system $\widetilde{\mathcal{F}_b}$ are strongly related to the fast basin of the original system $\mathcal{F}$.

Let $\mathcal{F} = (X; f_i, i = 1, \ldots, N)$ be an IFS with a strict attractor A. The *fast basin* of A is defined via

$$\widehat{B} = \{x \in X : F^k(\{x\}) \cap A \neq \emptyset\}.$$

Basic properties of fast basins are determined in [3].

Proposition 2 (A necessary condition for a homoclinic attractor)**.** *Let A be an attractor of $\mathcal{F}$ with the fast basin $\widehat{B}$. If A is an attractor of the modified system $\widetilde{\mathcal{F}_b}$, then $b \notin \widehat{B}$.*