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On the irrationality exponent of the generating function for a class of integer sequences☆

Min Niu*, Miaomiao Li

Department of Mathematics, University of Science and Technology Beijing, Beijing 100083, P.R. China

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ABSTRACT

We derive the upper bound of the irrationality exponent for a class of integer sequences with an assumption on their generating functions. If their Hankel determinants are weakly non-vanishing, then we prove that $(2\log b - 2\log|a|)/(\log b - 2\log|a|)$ is an upper bound of the irrationality exponent, where $a \in \mathbb{Z}/\{0\}$ and $b \in \mathbb{N}$ satisfying gcd $(a, b) = 1$ and $b > a^2$. On the other hand, by the classical technique from Diophantine approximation and the structure of generating function, we achieve an upper bound of the irrationality exponent for the 3-fold Morse sequence, whose Hankel determinants are not well studied.

1. Introduction

For a real irrational number ξ, let us recall that the irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ such that the inequality $|\xi - p/q| < 1/q^{\mu}$ has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. We remark that the irrationality exponent is also called the irrationality measure. It is well-known that $\mu(\xi) \geq 2$ for any irrational number ξ, and from Roth's theorem [8], one deduces that $\mu(\xi) = 2$ for any algebraic irrational number.

For each integer sequence $\mathbf{t} = \{t_i\}_{i \geqslant 0}$, let us define its generating function $T(\mathbf{t}, x)$ as the following:

$$T(\mathbf{t}, x) := \sum_{i=0}^{\infty} t_i x^{i-1}, \quad 0 < |x| < 1.$$

For simplicity, we write $T(x)$ instead of $T(\mathbf{t}, x)$ without danger of confusion.

In this paper, we will study the irrationality exponents for the class of integer sequences $\mathbf{t}$, whose generating functions satisfy the following assumption **A**:

The generating function $T(x)$ of the integer sequence $\mathbf{t}$ has the form

$$T(x) = \frac{A(x)}{B(x)} + C(x)T(x^k) \ (k \geq 2)$$

for some $A(x)$, $B(x)$, $C(x) \in \mathbb{Z}[x]$. As we will see, the Thue–Morse sequence, paper-folding sequence, the Cantor sequence and the 3-fold Morse sequence satisfy the assumption **A**.

The irrationality exponents of integer sequences have attracted a great deal of attention recently. However, in general, it is very difficult to determine the irrationality exponents, even for their upper bounds. Recently, a few papers on upper bounds of irrationality exponents for different sequences have been published, which are very important to promote the study of the irrationality exponents. For example, for the Thue–Morse sequence, Dubickas [4] in 2014 proved that its irrationality exponent $\mu(T(a/b))$ does not exceed $(2\log b - 2\log|a|)/(\log b - 2\log|a|)$ for any coprime nonzero integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ satisfying $b > a^2$. But even

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* Corresponding author. Tel.: +86 13240952786.
E-mail address: niuminfly@sohu.com (M. Niu).

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for the Thue–Morse, the value of the irrationality exponent is known only for the special case $a = \pm 1$. In fact, Bugeaud [3] in 2011 proved that $\mu(T(1/b)) = 2$, which improved the known upper bound $\mu(T(1/b)) \leq 4$ obtained in 2009 [1]. Therefore, for general integer sequences, it is worthy to derive the upper bounds of irrationality exponents.

As mentioned above, the main goal of this paper is to give an upper bound of the irrationality exponent of the generating function with the assumption **A**. For our purpose, two different methods will be introduced. One is the method based on Hankel determinants and the other is the Diophantine approximation.

One of our targets is to present an upper bound of the irrationality exponent $\mu(T(a/b))$ under the weakly non-vanishing property of Hankel determinants, see Definition 2.1 and Theorem 2.4 for details. It is well known that Hankel determinants of integer sequence are very important in the study of irrationality exponents. To review the researches on Hankel determinants and their applications, let us recall the definition of Hankel determinants according to our goals. Let $\mathbf{u} = \{u_k\}_{k\geqslant 0}$ be a sequence of complex numbers. Then the following (p, n)-order matrix $U_n^p(\mathbf{u})$ is called the Hankel matrix of $\mathbf{u}$.

$$U_n^p(\mathbf{u}) = \begin{bmatrix} u_p & u_{p+1} & \cdots & u_{p+n-1} \\ u_{p+1} & u_{p+2} & \cdots & u_{p+n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p+n-1} & u_{p+n} & \cdots & u_{p+2n-2} \end{bmatrix},$$

where $n \in \mathbb{N}$ and $p = 0, 1, 2, \ldots$. The determinant of the matrix $U_n^0(\mathbf{u})$ is called the Hankel determinant of the sequence $\mathbf{u}$ and we denote it by $H_n(\mathbf{u})$ in this paper. Allouche et al. [2] showed that all the Hankel determinants of the Thue–Morse sequence are nonzero, which implies that the Hankel determinants of the Thue–Morse sequence are weakly non-vanishing. Therefore, we generalize one of the main results obtained by Dubickas in [4]. In addition, as we see in Corollary 2.6, our result can also be applied to the paper-folding sequence and the Cantor sequence. In fact, for any nonzero integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ with $\gcd(a, b) = 1$ and $b > a^2$, we obtain the upper bound of irrationality exponents for the paper-folding sequence and the Cantor sequence. For the special case $a = 1$, they are studied recently in [5] and [9] respectively. In fact, in 2014, Guo et al. [5] calculated the Hankel determinants of the regular paper-folding sequence and obtained that the irrationality exponent of the regular paper-folding number is equal to 2. Wen and Wu [9] obtained the recurrence equations of the Hankel determinants associated with the Cantor sequence, and then with the help of the Hankel determinants, they proved that the irrationality exponent of the Cantor number is also equal to 2.

The other of our targets is to study the irrationality exponent relative to the 3-fold Morse sequence. Although Hankel determinants are widely studied and applied, there are much more integer sequences whose Hankel determinants have not been well studied. As we know, it seems that there is no result on the Hankel determinants for the 3-fold Morse sequence. Therefore, a different method is required for our goal. In Section 3, using classical technique from Diophantine approximation and the structure of generating function of this sequence, we achieve an upper bound of irrationality exponent of it.

This paper is organized as follows. In Section 2, we first introduce some important and essential lemmas, and then we estimate the upper bound of the irrationality exponent of the integer sequence with weakly non-vanishing Hankel determinants. In Section 3, we prove that the irrationality exponent of the generating function relative to the 3-fold Morse sequence is bounded from above by 6.

2. Upper bound for integer sequences with weakly non-vanishing Hankel determinants

In this section, we mainly study the upper bound of the irrationality exponent relative to the integer sequence $\mathbf{t}$, whose Hankel determinants are weakly non-vanishing, see Definition 2.1 below for its meaning. As the preparations for the proof of Theorem 2.4 below, let us first state some important lemmas.

Lemma 2.1 (See [1]). *Let ξ, δ, ρ and θ be real numbers such that $0 < \delta \leq \rho$ and $\theta \geq 1$. Let us assume that there exists a sequence $\{p_n/q_n\}_{n\geq 1}$ of rational numbers and some positive constants c_0, c_1 and c_2 such that*

(i) $q_n < q_{n+1} \leqslant c_0 q_n^{\theta}$;

(ii) $\dfrac{c_1}{q_n^{1+\rho}} \leqslant |\xi - \dfrac{p_n}{q_n}| \leqslant \dfrac{c_2}{q_n^{1+\delta}}$.

Then we have

$$\mu(\xi) \leqslant (1+\rho)\theta/\delta.$$

To formulate the next lemma and our main theorem precise, let us state the following definition.

Definition 2.1. Let $H_n(\mathbf{t})$ denote the Hankel determinants of the integer sequence $\mathbf{t}$. We say that the Hankel determinants $H_n(\mathbf{t})$ are weakly non-vanishing if there exits an increasing positive integer sequence $\{n_i\}_{i\geq 0}$ such that for all $i \geq 0$,

$$H_{n_i}(\mathbf{t}) H_{n_{i+1}}(\mathbf{t}) \neq 0.$$

Lemma 2.2 (See [5]). *Suppose the integer sequence $\mathbf{t}$ satisfies the assumption **A** and let $\deg(A(x)) = \alpha$, $\deg(B(x)) = \beta$, $\deg(C(x)) = \gamma$. If further the Hankel determinants $H_n(\mathbf{t})$ are weakly non-vanishing, then for each $l \in \mathbb{N}$ and sufficiently large m, there exist polynomials $P_{l,m}(x) \in \mathbb{Z}[x]$ of degrees at most $(\alpha+\beta+\gamma+l)k^m$, $Q_{l,m}(x) \in \mathbb{Z}[x]$ of degrees at most $(\beta+l)k^m$ and some positive constants $c_3(l)$, $c_4(l)$ such that*

$$c_3(l)(x^{Y_l k^m})^{\rho_l} \leqslant \left| T(x) - \frac{P_{l,m}(x)}{Q_{l,m}(x)} \right| \leqslant c_4(l)(x^{Y_l k^m})^{\delta_l}, \quad x \in \left(0, \frac{1}{2}\right], \tag{2.1}$$

where $Y_l = \alpha+\beta+\gamma+l$, $\rho_l = (2l+\gamma)/Y_l$ and $\delta_l = 2l/Y_l$.

Instead of the requirement that all the Hankel determinants of integer sequence are nonzero, i.e. the non-vanishing of Hankel determinants of integer sequence, the above lemma only requires weakly non-vanishing property of Hankel determinants $H_n(\mathbf{t})$, which is very important for our study in this section.

Lemma 2.3 (See [5]). *Let L, k, $m_0 \geq 2$ be positive integers, $R := R(L)$ be a real number. Let $\mathscr{B}$ be any subset of integers of $[k^{L-1}, k^L - 1]$ satisfying $[k^{L-1}, k^L - 1] \subset \bigcup_{x\in\mathscr{B}}[x-R, x+R]$.*