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Using spectral element method to solve variational inequalities with applications in finance

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ABSTRACT

Under the Black–Scholes model, the value of an American option solves a time dependent variational inequality problem (VIP). In this paper, first we discretize the variational inequality of American option in temporal direction by applying the Rannacher time stepping and achieve a sequence of elliptic variational inequalities. Second we discretize the spatial domain of variational inequalities by using spectral element methods with high order Lagrangian polynomials introduced on Gauss–Legendre–Lobatto points. Also by computing integrals by the Gauss–Legendre–Lobatto quadrature rule we derive a sequence of the linear complementarity problems (LCPs) having a positive definite sparse coefficient matrix. To find the unique solutions of the LCPs, we use the projected successive over-relaxation (PSOR) algorithm. Furthermore we present some existence and uniqueness theorems for the variational inequalities and LCPs. Finally, theoretical results are verified on the relevant numerical examples.

1. Introduction

Options are one of the most important types of financial instruments. Valuation of options has been a topic of research for more than three decades [17,32]. Black and Scholes proposed a model to transform the option pricing problem into the task of solving a parabolic partial differential equation (PDE) with a final condition [5]. Under the Black–Scholes model, price of an "American put" option must satisfy the following PDE:

$$\begin{aligned} &P_\tau(S,\tau) + \mathcal{L}_{BS}P(S,\tau) = 0, \quad S_f(\tau) < S < \infty, \quad 0 \le \tau < T, \\ &P(S,T) = h(S), \\ &P(S_f(\tau),\tau) = h(S), \quad \frac{\partial P}{\partial S}(S_f(\tau),\tau) = -1, \\ &\lim_{S\to\infty} P(S,\tau) = 0, \end{aligned} \tag{1.1}$$

where $h(S) = \max\{E - S, 0\}$ and $\mathcal{L}_{BS}$ is the Black–Scholes partial differential operator introduced by

$$\mathcal{L}_{BS}P(S,\tau) = \frac{1}{2}\sigma^2 x^2 P_{SS}(S,\tau) + rSP_S(S,\tau) - rP(S,\tau).$$

In (1.1), $P(S, \tau)$ denotes the price of option at time τ when the spot price of underlying asset is S. The "strike price" E, "volatility" σ, "interest rate" r and "time to maturity" T are all positive constants. The final condition $h(S) = \max\{E - S, 0\}$ is the value of option at time $t = T$ and usually called the "pay off" function. Eq. (1.1) is a parabolic PDE with free boundary conditions. We need to find the solution $P(S, \tau)$ and the unknown "free boundary" $S_f(\tau)$. It can be proven that the "optimal exercise boundary" $S_f(\tau)$ for American put options is a monotonic non-decreasing function of τ, satisfying the following expression (see [19])

$$S_{min} < S_f(\tau) \le E, \quad S_{min} = \frac{E}{1-\frac{1}{\lambda}},$$
$$\lambda = \rho - \sqrt{\rho^2 + \frac{2r}{\sigma^2}}, \quad \rho = \frac{-r + \frac{\sigma^2}{2}}{\sigma^2}. \tag{1.2}$$

In Fig. 1, we present the free boundary $S_f(\tau)$ in the semi infinite domain of problem (1.1). Many problems in physics, industry, finance, and other areas can be described by free

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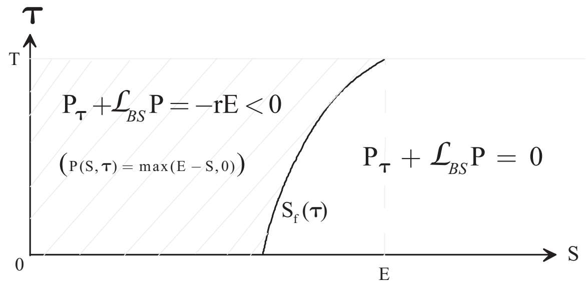

Fig. 1. The semi infinite domain of the Black–Scholes equation and the optimal exercise boundary $S_f(\tau)$. In the right hand side of $S_f(\tau)$, the differential equation is hold. In the left hand side of curve $S_f(\tau)$, the solution $P(S, \tau)$ coincides with the plane $h(S) = \max\{E - S, 0\}$.

boundary problems. The theory of free boundary problems is closely related to VIPs and LCPs which [8,10,11]. It is possible to formulate the free boundary problem (1.1) as a LCP (see e.g. [9,27]). To this end we notice that if $S_f(\tau) < S < \infty$, the first equation in (1.1) holds. For $0 \le S \le S_f(\tau)$, the solution $P(S, \tau)$ coincides with the plane $h(S) = \max\{K - S, 0\}$. So we have $P_\tau(S, \tau) + \mathcal{L}_{BS}P(S, \tau) = -rE < 0$. In summary, on the entire half strip, $P(S, \tau)$ must satisfy an inequality of the Black–Scholes type: $P_\tau(S, \tau) + \mathcal{L}_{BS}P(S, \tau) \le 0$. Furthermore for financial reasons [17], an American option can not have a value that is smaller than the pay off function. Therefore in the entire domain we have $P_\tau(S, \tau) \ge h(S)$ (the equality is occur if and only if $0 \le S \le S_f(\tau)$). So in the entire domain we have $(P - h) = 0$ or $P_\tau(S, \tau) + \mathcal{L}_{BS}P(S, \tau) = 0$. In conclusion Eq. (1.1) can be formulated as

$$\begin{cases} P_\tau(S, \tau) + \mathcal{L}_{BS}P(S, \tau) \le 0, & (S, \tau) \in (0, \infty) \times (0, T), \\ P_\tau(S, \tau) \ge h(S), & \\ (P - h)(P_\tau(S, \tau) + \mathcal{L}_{BS}P(S, \tau)) = 0, & \\ P(0, \tau) = E, \quad \lim_{S\to\infty} P(S, \tau) = 0. & \end{cases} \tag{1.3}$$

Notice that the unknown boundary $S_f(\tau)$ does not occur in (1.3) explicitly and we have to find only the solution $P(S, \tau)$. The closed form solution to the problem (1.1) or (1.3) does not exist and the solution has to be computed numerically. So far a variety of numerical methods have been developed for solving the free boundary problem (1.1) and the equivalent problem (1.3). For example, the finite difference and finite element methods were proposed to solve (1.1) and (1.3) [2,9,29,31]. A comparison of several numerical methods for valuation American options is provided in [4]. The most widely numerical technique used for pricing American options is the “binomial method” proposed by Cox et al. [7]. A comprehensive algorithmic description and implementation of binomial method were given in [15].

In this paper, we discretize (1.1) by spectral element methods proposed by [25]. Spectral element methods combine the Galerkin spectral methods with finite element methods by applying the spectral method per element. The basis functions in this method usually are Lagrangian polynomials introduced on Gauss–Legendre–Lobatto points. Performing Gauss–Legendre–Lobatto integration rule leads to a diagonal mass matrix which decreases the cost of computations. The spectral element methods recently have been applied to solve a wide range of problems in science and engineering (see [20,22,23,28]).

The remainder of this paper is organized as follows. In Section 2, we review some definitions, properties and theorems for variational inequalities and LCPs. We investigate the one dimensional obstacle problem as a simple example of variational inequalities in Section 3. In Section 4, we formulate problem (1.3) as an VIP, then we approximate the unique solution of the problem by using spectral element methods. We give two illustrative examples to demonstrate the validity and applicability of the proposed methods in Section 5. Section 6 consists of a brief conclusion.

2. Variational inequality problems

Many mathematical problems can be formulated as VIPs introduced by Hartman and Stampacchia [14]. In this section we review some properties of variational inequalities and LCPs in the Hilbert spaces.

2.1. Variational inequality problems in the Hilbert spaces

Suppose that V is a Hilbert space with scalar product $(.,.)$ and associated norm $\|.\|_V$. Let

- $a(.,.) : V \times V \to \mathbb{R}$ be a bilinear form on $V \times V$, that is continuous i.e. there exists a constant C such that $|a(u, v)| \le C\|u\|_V\|v\|_V$, $\forall u, v \in V$.
- $L : V \to \mathbb{R}$ be a continuous linear functional,
- $\mathcal{K}$ be a closed convex nonempty subset of V.

A variational inequality problem in Hilbert space V can be presented as the following problem:

Problem. (Variational inequality problem in Hilbert spaces)

$$\text{Find } u \in \mathcal{K} \quad \text{such that} \quad a(u, v - u) \ge L(u, v - u), \quad \forall v \in \mathcal{K}. \tag{2.1}$$

The following theorem is a generalizations of the Lax–Milgram theorem and provides a sufficient condition for existence and uniqueness of problem (2.1) (see [13, Theorem 3.1]).

Theorem 2.1 (Stampaccia). *Let $a(.,.)$ be coercive i.e. there exists a positive constant α such that*

$$a(v, v) \ge \alpha\|v\|^2, \quad \forall v \in V,$$

then problem (2.1) has a unique solution.

Remark 2.2. Let the bilinear form $a(u, v)$ be an inner product such that its induced norm be equivalent to $\|.\|_V$. Then it is easy to see that $a(.,.)$ is continuous and coercive, so the problem (2.1) has a unique solution.

2.2. Variational inequalities and LCPs in the finite dimensional Euclidian spaces

Discretization of variational inequality problem (2.1) leads to a variational inequality in a finite dimensional Euclidian space.