Chaos, Solitons and Fractals 81 (2015) 290–298

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Nonlocal symmetry analysis, explicit solutions and conservation laws for the fourth-order Burgers' equation

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ARTICLE INFO

Article history:
Received 16 June 2015
Accepted 27 September 2015

MSC:
22E70
35Qxx
70S10

PACS:
11.30.-j
02.20.-a
02.20.Sv
02.30.Jr

Keywords:
Fourth-order Burgers' equation
Nonlocal symmetry analysis
Linearization
Explicit solutions
Conservation laws

ABSTRACT

The Painlevé analysis is carried out on the physical form of the fourth order Burgers' equation. Then, nonlocal symmetries of the equation are constructed. Also, linearizations are derived based on the symmetries. In particular, the explicit solution of the equations are presented in terms of Hopf–Cole transformations. Furthermore, nonlinear self-adjointness and conservation laws of potential equation are investigated with symmetries.

1. Introduction

In this paper, we study the following equation

$$u_t = u_{xxxx} + 10u_xu_{xx} + 4uu_{xxx} + 12uu_x^2 + 4u^3u_x + 6u^2u_{xx}. \quad (1)$$

This equation is derived from the following Burgers' hierarchy [1–7]

$$u_t = K_m(u) = (D + u + u_xD^{-1})^{m-1}u_x, \quad m = 1, 2, \ldots \quad (2)$$

The hierarchy (2) contains many equations, such as the seminal Burgers' equation

$$u_t = 2uu_x + u_{xx}, \quad (3)$$

the famous STO equation (the third order Burgers' equation)

$$u_t = 3uu_{xx} + u_{xxx} + 3u_x^2 + 3u^2u_x, \quad (4)$$

and so on.

It is known that Burgers' equation is a fundamental model from fluid mechanics. In 1939, Burgers [1], first introduced this equation, simplified the Navier–Stokes equation by just dropping the pressure term. It appears in various fields of science, such as modeling of gas dynamics and traffic flow and so on. It is also used for describing wave processes in

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http://dx.doi.org/10.1016/j.chaos.2015.09.030
0960-0779/

acoustics and hydrodynamics. These types of Burgers' equation have been studied by many authors in last decades ([1–18] and papers cited therein).

Symmetries (local and nonlocal) and conservation laws (CLs) (local and nonlocal) play key roles in the study and development of nonlinear science, in particular in mathematics and physics. Group method [8–11] provide a systemic method to deal with differential equations. For partial differential equations (PDEs), it can reduce the number of variables PDEs; From ordinary differential equations (ODEs) point of view, it can reduce the order of ODEs. CLs describe physical conserved quantities such as mass, energy, momentum and others. In addition, CLs also help the differential equations in the development of numerical methods.

In order to provide more background knowledge about Burgers' type of equation, it is necessary to study the high order Burgers' equation. In this paper, the nonlocal symmetry and explicit solutions of the forth order Burgers' equation are constructed. Furthermore, the nonlinear self-adjointness and conservation laws of the potential equation are derived via symmetries. The paper is divided as follows. In Section 2, the truncated Painlevé analysis of the equation are derived. In Section 3, nonlocal symmetries, linearization and explicit solutions are constructed. In Section 4, potential equation are investigated. Based on the symmetries, nonlinear self-adjointness and conservation laws for the potential equation are presented. Finally, the obtained results and some concluding remarks are given in Section 5.

2. Painlevé analysis

We first begin with the following Laurent series for $u(x, t)$

$$u = \sum_{j=0}^{\alpha} u_j \phi^{j+\alpha}, \tag{5}$$

with a sufficient number of arbitrary functions and related to derivatives of ϕ. In addition, the leading orders of $\alpha < 0$, that is to say, α should be negative integers. Putting the following term ($j = 0$)

$$u = u_0 \phi^{-\alpha}, \tag{6}$$

into Eq. (1), balancing of the dominant terms determines, and analyzing leading order, one can derive $\alpha = -1$, and $u_0 = \phi_x$, $u_0 = 2\phi_x$, or $u_0 = 3\phi_x$.

For the case $u_0 = \phi_x$, plugging Eq. (5) into Eq. (1), we get

$$\begin{aligned}\phi_x^4 u_j (j+1)(j-1)(j-2)(j-4) \\ = F(u_{j-1} \cdots u_0, \phi_t, \phi_{xx}, \ldots).\end{aligned} \tag{7}$$

It is found that the resonances appear at

$$j = -1, 1, 2, 4. \tag{8}$$

This equation possesses the Painlevé property at $j = 1, 2, 4$. To look for a Bäcklund transform and get a finite expansion, assume that

$$u_2 = u_3 = u_4 = 0, \tag{9}$$

and we get the the following Bäcklund transform

$$u = \frac{u_0}{\phi} + u_1. \tag{10}$$

In particular, when $u_1 = 0$, the famous Cole–Hopf transform

$$u = \frac{\phi_x}{\phi}, \tag{11}$$

is obtained from Eq. (10). Also, we arrive at

$$\phi_t = \phi_{xxxx}. \tag{12}$$

Moreover, it is noted that $u_0 = \phi_x$ is a nonlocal symmetry with a solution of u_1 as well. On the other hand, putting Eq. (11) into Eq. (1), then we can get

$$\frac{\phi_{tx} - \phi_{xxxxx}}{\phi} + \frac{\phi_t \phi_x - \phi_x \phi_{xxxx}}{\phi^2} = 0, \tag{13}$$

that is to say, $\phi_t = \phi_{xxxx}$. In other words, on the basis of Hopf–Cole transformation, the Eq. (1) can transform into a forth order linear Eq. (11). In Section 2, we will display that Eq. (1) also transformed into the same fourth order linear equation Eq. (11) based on the symmetries.

3. Nonlocal symmetries, linearization and explicit solutions

3.1. Nonlocal symmetries of the potential system

Seeking nonlocal symmetries of PEDs is an important work. As nonlocal symmetries involved in new variables, and maybe can get new solutions and nonlocal conservation laws. The authors obtained infinite many nonlocal symmetries by inverse recursion operators [18,19], the conformal invariant form (Schwartz form) [20], Darboux transformation [21,22], Lax pair [23] and so on. The authors employed the potential symmetry to derive nonlocal symmetries [8]. In order to get nonlocal symmetries of Eq. (1), we present potential symmetry analysis to handle Eq. (1). First, we rewrite Eq. (1) in the following potential system

$$\begin{aligned} v_x &= u, \\ v_t &= u_{xxx} + u^4 + 6u^2 u_x + 4u u_{xx} + 3u_x^2, \end{aligned} \tag{14}$$

and substitute the first equation to the second equation, one can get the following potential equation

$$v_t = v_{xxxx} + (v_x)^4 + 6v_x^2 v_{xx} + 4v_x v_{xxx} + 3v_{xx}^2. \tag{15}$$

Consider the point symmetry

$$\begin{aligned} t^* &= t + \epsilon\tau(x, t, u, v) + O\big(\epsilon^2\big), \\ x^* &= x + \epsilon\xi(x, t, u, v) + O\big(\epsilon^2\big), \\ u^* &= u + \epsilon\eta(x, t, u, v) + O\big(\epsilon^2\big), \\ v^* &= v + \epsilon\psi(x, t, u, v) + O\big(\epsilon^2\big), \end{aligned} \tag{16}$$

it holds the following equations

$$\begin{aligned} &X^{(3)}(v_x - u) = 0, \\ &X^{(3)}(v_t - u_{xxx} - u^4 - 6u^2 u_x - 4u u_{xx} - 3u_x^2) = 0, \end{aligned} \tag{17}$$

for any (u, v). Furthermore, infinitesimal generator is given by

$$\begin{aligned} X = {}& \tau(x, t, u, v)\frac{\partial}{\partial t} + \xi(x, t, u, v)\frac{\partial}{\partial x} \\ &+ \eta(x, t, u, v)\frac{\partial}{\partial u} + \psi(x, t, u, v)\frac{\partial}{\partial v}. \end{aligned} \tag{18}$$