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Block implicit Adams methods for fractional differential equations

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ABSTRACT

In this paper, we present a family of Implicit Adams Methods (IAMs) for the numerical approximation of Fractional Initial Value Problems (FIVP) with derivatives of the Caputo type. A continuous representation of the k-step IAM is developed via the interpolation and collocation techniques and adapted to cope with the integration of FIVP. This is achieved by combining the k-step IAM with $(k-1)$ additional methods obtained from the same continuous scheme and applying them as numerical integrators in a block-by-block fashion. We also investigate the stability properties of the block methods and the regions of absolute stability of the methods are plotted in the complex plane. The block methods are tested on numerical examples including large systems resulting from the semi-discretization of one-dimensional fractional heat-like partial differential equations.

1. Introduction

Fractional differential equations (FDEs) arise in the mathematical modelling of several physical phenomena and play an important role in various branches of science and engineering. Applications of FDEs are found in chemistry, electronics, circuit theory, seismology, signal processing, control theory and so on. Also, these FDEs serve as a generalization of their corresponding ordinary differential equations (ODEs). For a brief history and introduction to fractional calculus, we refer the reader to [15–17].

In what follows, we consider the FIVP in the form

$$\begin{aligned} {}_cD^{\alpha}_{x_0} y(x) &= f(x, y(x)) \\ y(x_0) &= y_0 \end{aligned} \qquad (1)$$

where $0 < \alpha < 1$ is the fractional order and ${}_cD^{\alpha}_{x_0}$ (in the sequel we shall simply use D^{α}) denotes the Caputo α derivative operator which is defined as

$$D^{\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-s)^{-\alpha} y'(s) ds \qquad (2)$$

We have adopted the Caputo's definition of derivatives of non integer order (which is a modification of the Riemann–Liouville definition) since it can be coupled with initial conditions having a clear physical meaning. The existence and the uniqueness of the solution of (1) has been given in Deithelm and Ford [2].

Due to the occurrence of FDEs in several models, there have been an increasing attention for the development of effective and well suited methods for this class of important problems. Several methods have been proposed and analysed for the numerical approximation of FDEs (See Lubich [11–13] , Garrapa [7], Galeone and Garrapa [6,8,9], Diethelm et al. [3,4] and the references therein). These authors have independently developed Fractional Linear Multistep Methods (FLMMs) using convolution quadratures. Diethelm et al. [3,4] used the rectangle rule and

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the two-point trapezoidal quadrature formula to derive explicit one-step Adams–Bashforth method and implicit one-step Adams–Moulton method, respectively with the former method serving as a predictor for the later. Lubich [12] also proposed formulas of the form

$$y_n = f(t_n) + h^\alpha \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} g(t_j, y_j) + h^\alpha \sum_{j=0}^{m} \omega_{nj} g(t_j, y_j), \quad nh \in \mathbb{I} \tag{3}$$

where ω_n^α and ω_{nj} are the convolution and starting quadrature weights respectively and are independent of the step size h.

One major difficulty in the FLMMs (3) is in evaluating the convolution weights ω_n^α. Most of the methods rely on the J.C.P Miller formula for the computation of these weights. In order to avoid this major drawback, we give a different approach in the construction of the FLMMs. This approach is based on interpolation and collocation as was discussed by Onumanyi et al. [18].

The paper is organized as follows: In Section 2, we discuss the development of the Fractional Adams Moulton's Method. Section 3 details the stability properties and implementation of the methods. In Section 4, we give five test problems to elucidate our theoretical results. Finally, we give some concluding remarks in Section 5.

2. Fractional Adams Moulton's methods

In this section, we shall construct a k-step Continuous Fractional Adams Moulton's Methods (CFAMM) which will be used to obtain the discrete Fractional Adams Moulton's Methods (FAMM). The CFAMM has the general form

$$U(x) = \gamma_{k-1}(x) y_{n+k-1} + h^\alpha \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{4}$$

where $\gamma_{k-1}(x), \beta_j(x)$ are continuous coefficients. We assume that $y_{n+j} = U(x_n + jh)$ is the numerical approximation to the analytical solution $y(x_{n+j})$ and $f_{n+j} = D^\alpha U(x_n + jh)$ is an approximation to $D^\alpha y(x_{n+j})$. The CFAMM is constructed from its equivalent form by requiring that the exact solution $y(x)$ is locally approximated by the function (4) on the interval $[x_n, x_{n+k}]$.

Theorem 2.1. *Let (4) satisfy the following conditions*

$$\begin{aligned} U(x_{n+k-1}) &= y_{n+k-1} \\ D^\alpha U(x_{n+j}) &= f_{n+j}, \quad j = 0(1)(k) \end{aligned} \tag{5}$$

then the continuous representation (4) is equivalent to

$$U(x) = \sum_{j=0}^{k} \frac{det(V_j)}{det(V)} P_j(x) \tag{6}$$

where we define the matrix V as

$$V = \begin{pmatrix} P_0(x_{n+k-1}) & \cdots & P_k(x_{n+k-1}) \\ D^\alpha P_0(x_n) & \cdots & D^\alpha P_k(x_n) \\ D^\alpha P_0(x_{n+1}) & \cdots & D^\alpha P_k(x_{n+1}) \\ \vdots & \vdots & \vdots \\ D^\alpha P_0(x_{n+k}) & \cdots & D^\alpha P_k(x_{n+k}) \end{pmatrix},$$

V_j is obtained by replacing the jth column of V by W where T denotes the transpose, $P_j(x) = x^j$, $j = 0(1)k$ are basis functions and W is a vector given by

$$W = (y_{n+k-1}, f_n, f_{n+1}, \ldots, f_{n+k})^T.$$

Proof. We require that the method (4) be defined by the assumed polynomial basis functions

$$\begin{aligned} \gamma_{k-1}(x) &= \sum_{i=0}^{k} \gamma_{i+1,k-1} P_i(x) \\ h^\alpha \beta_j(x) &= \sum_{i=0}^{k} h^\alpha \beta_{i+1,j} P_i(x), \quad j = 0(1)k \end{aligned} \tag{7}$$

where $\gamma_{i+1,k-1}, h^\alpha \beta_{i+1,j}$ are coefficients to be determined.

Substituting (7) into (4), we have

$$U(x) = \sum_{i=0}^{k} \gamma_{i+1,k-1} P_i(x) y_{n+k-1} + \sum_{j=0}^{k} \sum_{i=0}^{k} h^\alpha \beta_{i+1,j} P_i(x) f_{n+j}$$

which may be written as

$$U(x) = \sum_{i=0}^{k} \left\{ \gamma_{i+1,k-1} y_{n+k-1} + \sum_{j=0}^{k} h^\alpha \beta_{i+1,j} f_{n+j} \right\} P_i(x)$$

and expressed as

$$U(x) = \sum_{i=0}^{k} \tau_i P_i(x) \tag{8}$$

where

$$\tau_i = \gamma_{i+1,k-1} y_{n+k-1} + \sum_{j=0}^{k} h^\alpha \beta_{i+1,j} f_{n+j}$$

By imposing condition (5) on (8), we obtain a system of $(k+1)$ equations, which can be expressed as $V = LW$ where $L = (\tau_0, \tau_1, \ldots, \tau_k)^T$ is a vector of $(k+1)$ undetermined coefficients. Using Crammer's rule, the elements of L can be obtained and are given by

$$\tau_i = \frac{det(V_j)}{det(V)}, \quad j = 0(1)k$$

where V_j is obtained by replacing the jth column of V by W. We rewrite (8) using the newly found elements of L as

$$U(x) = \sum_{j=0}^{k} \frac{det(V_j)}{det(V)} P_j(x) \tag{9}$$

□

Remark 2.2. The continuous scheme (4) which is equivalent to (6) is evaluated at x_{n+k} to obtain the k-step FAMM of the form

$$y_{n+k} - y_{n+k-1} = h^\alpha \sum_{j=0}^{k} \beta_j f_{n+j} \tag{10}$$

Also, we emphasize that the continuous scheme (6) is evaluated at x_{n+i}, $i = 0(1)(k-2)$ to obtain