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Equivariant characteristic forms on the bundle of connections

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Abstract

The characteristic forms on the bundle of connections of a principal bundle $P \to M$ of degree equal to or less than $\dim M$, determine the characteristic classes of P, and those of degree $k + \dim M$ determine certain differential k-forms on the space of connections $\mathcal{A}$ on P.

The equivariant characteristic forms provide canonical equivariant extensions of these forms, and therefore canonical cohomology classes on $\mathcal{A}/\mathrm{Gau}^0 P$. More generally, for any closed $\beta \in \Omega^r(M)$ and $f \in \mathcal{I}_k^G$, with $2k + r \geq \dim M$, a cohomology class on $\mathcal{A}/\mathrm{Gau}^0 P$ is obtained. These classes are shown to coincide with some classes previously defined by Atiyah and Singer.

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1. Introduction

Let $\pi : P \to M$, be a principal G-bundle and let $p : C(P) \to M$ be its bundle of connections. Let $\mathcal{I}_k^G$ be the space of Weil polynomials of degree k for G. The principal G-bundle

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$C(P)\times_M P\to C(P)$ is endowed with a canonical connection $\mathbb{A}$ (see below for the details), which can be used to obtain, for every $f\in\mathcal{I}_k^G$, a characteristic $2k$-form on $C(P)$, denoted by $c_f(\mathbb{F})=f(\mathbb{F},\ldots^{(k},\mathbb{F})$ (e.g., see [11]), where $\mathbb{F}$ is the curvature of $\mathbb{A}$. Moreover, such a form is closed and Aut P-invariant. As $C(P)$ is an affine bundle, the map $p^*: H^*(M)\to H^*(C(P))$ is an isomorphism. The cohomology class in M corresponding to $c_f(\mathbb{F})$ under this isomorphism is the characteristic class of P associated to f. Hence, the characteristic forms on $C(P)$ determine the characteristic classes on M, but the characteristic forms contain more information than the characteristic classes; for example, the characteristic classes of degree $2k>n$ vanish, although the corresponding forms do not necessarily, as $\dim C(P)>\dim M$. Precisely, the principal aim of this paper is to provide a geometric interpretation of such characteristic forms of higher degree.

This is based on the following construction. Let $E\to N$ be an arbitrary bundle over a compact, oriented n-manifold without boundary. We define a map $\mathcal{F}: \Omega^{n+k}(J^rE)\to\Omega^k(\Gamma(E))$ commuting with the exterior differential and with the action of the group $\mathrm{Proj}^+(E)$ of projectable diffeomorphisms which preserve the orientation on M. Hence, if $\alpha\in\Omega^{n+k}(J^rE)$ is closed, exact, or invariant under a subgroup $\mathcal{G}\subset\mathrm{Proj}^+(E)$, then the form $\mathcal{F}[\alpha]$ enjoys the same property.

Applying this construction to the bundle $C(P)\to M$, for any characteristic form $c_f(\mathbb{F})$ with $2k>n$, we obtain a closed and Gau P-invariant $(2k-n)$-form on the space $\mathcal{A}=\Gamma(M,C(P))$ of connections on P. More generally, as proved in [11], the space of Gau P-invariant forms on $C(P)$ is generated by forms of type $c_f(\mathbb{F})\wedge p^*\beta$, with $\beta\in\Omega^*(M)$. So, given $f\in\mathcal{I}_k^G$ and a closed $\beta\in\Omega^r(M)$, such that $2k+r\geq n$, we have a closed and Gau P-invariant $(2k+r-n)$-form on $\mathcal{A}$ given by

$$C_{f,\beta}=\mathcal{F}[c_f(\mathbb{F})\wedge p^*\beta]\in\Omega^{2k+r-n}(\mathcal{A}). \tag{1}$$

As $\mathcal{A}$ is an affine space, these forms are exact, and the cohomology classes defined by them on $\mathcal{A}$, vanish; but in gauge theories—because of gauge symmetry—it is more interesting to consider the quotient space $\mathcal{A}/\mathrm{Gau}\,P$ instead of the space $\mathcal{A}$ itself. Although the forms (1) are Gau P-invariant, they are not projectable with respect to the natural quotient map $\mathcal{A}\to\mathcal{A}/\mathrm{Gau}\,P$. Hence they do not define directly cohomology classes on $\mathcal{A}/\mathrm{Gau}\,P$. Consequently, we are led to consider another way in order to obtain cohomology classes on the quotient from these forms. As is well known, the cohomology of the quotient manifold by the action of a Lie group, is related to the equivariant cohomology of the manifold, e.g., see [19]. Below, we show that the usual construction of equivariant characteristic classes (e.g., see [6,7,9]) when applied to the canonical connection $\mathbb{A}$, provides canonical Aut P-equivariant extensions of the characteristic forms. By extending the map $\mathcal{F}$ to equivariant differential forms in an obvious way, this result allows us to obtain Gau P-equivariant extensions of the forms (1); see Theorem 16 below. These extensions determine cohomology classes in the quotient space $\mathcal{A}/\mathrm{Gau}^0 P$, where $\mathrm{Gau}^0 P\subset\mathrm{Gau}\,P$ is the subgroup of gauge transformations preserving a fixed point $u_0\in P$. We also prove that such classes coincide with those defined in [3].

As is well known (e.g. see [2]), an equivariant extension of an invariant symplectic two-form is equivalent to a moment map for it. Hence, if the form (1) is of degree two on $\mathcal{A}$, then the Gau P-equivariant extension that we obtain, defines a canonical moment map for