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Chaos, Solitons and Fractals 25 (2005) 449–460

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Characterizing chaotic processes that generate uniform invariant density

Weihong Huang

Nanyang Technological University, Nanyang Avenue, Nanyang 639798, Singapore

Accepted 23 November 2004

Communicated by Prof. S. Yousefi

Abstract

Universal formulations for four types of discrete chaotic processes that generate (preserve) uniform invariant density are provided. Characterizations such as necessary and/or sufficient conditions are established. It is revealed that such processes are "invariant" with branch-mirroring and horizontal mirroring. In addition, horizontally linear combinations of such processes remain to be in the same family. Theoretical findings are well verified by the computer simulations.

1. Introduction

Probabilistic or statistical approach is proved to be one of most effective approaches in the study of chaotic dynamical systems since it by-passes the limitations caused by "sensitivity" and provides us with various global and long-run properties of dynamical trajectories [1–4]. The key element of statistical approach is the invariant density that describes the "steady state" of a nonlinear system, into which the system settles after a long run of iterations starting from "almost anywhere" (in Lebesgue sense). The knowledge of invariant density is also essential for the calculation of relevant experimental oberservables such as time correlations and their power spectra [2].

While fruitful results were established in the existence of absolutely continuous invariant densities and their numerical approximations, the progress related to the dynamical systems with analytical invariant measures is still quite limited. Among the most challenging questions are (i) constructing an invariant measure (density) analytically for a given dynamical system and (ii) revealing the generic properties exhibited in a given class of chaotic systems that generate a specified invariant measure [5]. The current investigation intends to make an advance in the very latter direction by exploring the generic properties exhibited in the class of discrete dynamical systems (processes) that generate a uniform invariant density. The study is of theoretical significance due to the fact that any other discrete dynamic process is conjugated to one of such processes. It also relates to the extensive literatures regarding construction of random number generators [6].

E-mail address: awhhuang@ntu.edu.sg

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2004.11.016

The article is organized as follows. Section 2 defines four types of two-segmental Lebesgue processes, that is, the two-segmental processes that generate (preserve) invariant density. A necessary condition is provided for such process. Section 3 then provides a universal formulation of two-segmental Lebesgue processes. Geometrical characterization of two-segmental Lebesgue processes is illustrated in Section 4. Section 5 further reveals the fact that a horizontal mirroring of a two-segmental Lebesgue process is still a Lebesgue process. It is again verified in Section 6 that for any two Lebesgue processes that belong to the same type and have an identical turning point, their horizontally linear combinations belong to the same type of Lebesgue process. Section 7 concludes the paper.

2. Two-segmental Lebesgue processes

Without loss of generality, we confine ourselves to the discrete chaotic processes defined on a unit-interval $I \doteq [0,1]$. The particular choice of the unit-interval I as the domain of variable is not restrictive since, given any dynamical process, $x_{t+1} = S(x_t)$, where S is a mapping from an interval $[a,b]$ to $[a,b]$ with $S(a) = a$ and $S(b) = b$, we can always perform variable substitution

$$h(x_t) = a + (b-a)y_t,$$

so that the domain of the transformed dynamical system given by

$$y_{t+1} = s(y_t) = h^{-1}(S(h(y_t))) = \frac{S(a+(b-a)y_t) - a}{b-a}, \tag{1}$$

is restricted to the unit interval I, where $h^{-1} = (x - a)/(b - a)$ is the inverse-transformation of h. The new dynamical process (1) is conventionally referred to as a conjugate of S. It can be shown that the dynamical properties of two conjugated dynamical processes are qualitatively identical [1,6] in the sense that they have identical Lyapunov components and identical asymptotic properties.

A discrete dynamic process

$$x_{t+1} = f(x_t), \quad 0 \leqslant x_t \leqslant 1,$$

is said to be *two-segmental* if there exists such a turning point (partitioning point) $\hat{x} \in I$ that f can be expressed as

$$f(x) = (f_1(x), f_2(x)) = \begin{cases} f_1(x), & 0 \leqslant x \leqslant \hat{x}, \\ f_2(x), & \hat{x} \leqslant x \leqslant 1, \end{cases} \tag{2}$$

where $f_1{:}[0,\ \hat{x}] \to [0,1]$ and $f_2 : [\hat{x},1] \to [0,1]$ are monotonically continuous, differentiable except possibly at finite points, and onto in the sense of $f_1(0) \in \{0,1\}$, $f_2(1) \in \{0,1\}$, with $f_1(\hat{x}) = 1 - f_1(0)$ and $f_2(\hat{x}) = 1 - f_2(1)$.

For the convenience of late references, we shall refer f_1 as the left-branch of f and denote it as f_L if $f_1' \geqslant 0$ and as f_l if $f_1' \leqslant 0$, respectively. Similarly, f_2 will be referred to as the right-branch of f and denoted as f_R if $f_2' \geqslant 0$ and as f_r if $f_2' \leqslant 0$, respectively.

Fig. 1 illustrates four possible types of two-segmental full range discrete chaotic processes, which will be referred to as H-type, V-type, N-type, and S-type, and denoted as F_H, F_V, F_N, and F_S, respectively [1]. A two-segmental discrete process will be referred to as a *Lebesgue process* if it generates the uniform density. Lebesgue processes form an extremely important family because all two-segmental fully-developed (or complete) chaotic processes are conjugated to one member of such family [7].

The first step to characterize Lebesgue processes is to explore what are the necessary conditions for a two-segmental process to be a Lebesgue process. It turns out to be that, regardless of what type the process belongs to, the two branches must be steep enough in the sense that the absolute values of slopes must be greater than unity all over the domain.

Proposition 1. *The necessary conditions for a two-segmental process* $f = (f_1,f_2)$ *with* $f_1 \in \{f_L,f_l\}$ *and* $f_2 \in \{f_R,f_r\}$ *to be a Lebesgue process is*

$$|f'(x)| > 1 \quad \textit{for almost all } x \in I,$$

that is, the derivatives of all branches must exceed unity in absolute value.

[1] Here "H" stands for "hat", "S" for "sawtooth", while "N" and "V" symbolize the shape of respective maps.