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The Da Rios system under a geometric constraint: the Gilbarg problem

W.K. Schief*, C. Rogers

School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia

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Abstract

A classical problem in hydrodynamics originally posed by Gilbarg has been recently reduced to that of solving a solitonic Heisenberg spin equation subject to a geometric constraint. Here, this reformulation is shown to lead to a class of solutions of the Gilbarg problem corresponding to travelling wave solutions of a system derived by Da Rios in 1906.

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1. Introduction

A long-standing problem in hydrodynamics posed by Gilbarg [1] and subsequently investigated by Prim [2], Howard [3], Wasserman [4] and Marris [5] has recently been shown to be encapsulated in a nonlinear system consisting of an integrable Heisenberg spin

* Corresponding author. Tel.: +61 2 9385 7051; fax: +61 2 9385 7123.
E-mail address: schief@maths.unsw.edu.au (W.K. Schief).

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equation subject to a geometric constraint [6,7]. This Heisenberg spin equation is equivalent to the celebrated nonlinear Schrödinger (NLS) equation which, in turn, is a consequence of the classical Da Rios system which was originally set down in 1906 in connection with the spatial evolution of an isolated vortex filament in an unbounded inviscid liquid [8]. The hydrodynamics problem treated in [1–7], in fact, represents a generalisation of a well-known problem posed and resolved by Hamel [9]. An alternative demonstration of what has come to be known as Hamel's theorem has been given in [10] via a geometric formalism originally introduced by Marris and Passman [11] in a kinematic study of hydrodynamics. This formalism has been exploited in magnetohydrodynamics [12] and recently within the context of the geometry of soliton theory [13,14]. Here, it is used to address what we term the Gilbarg problem which seeks to delimit steady hydrodynamic motions for which the speed of the fluid flow is constant along streamlines. In view of the continuity equation, this condition is equivalent to the purely geometric constraint $\operatorname{div} \boldsymbol{t} = 0$, where $\boldsymbol{t}$ is the unit tangent to a generic streamline. It is established that for such motions, two important geometric constraints must apply on the abnormality $\Omega = \boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{t}$. Remarkably, it is demonstrated that these constraints encode the 'travelling wave' symmetry reduction of the Da Rios system if Ω is assumed to be constant on the constant pressure surfaces. This result encapsulates Hamel's theorem corresponding to $\Omega = 0$. The motions of Gilbarg type which are compatible with the travelling wave reduction of the Da Rios system have recently been delimited in [15].

It is of interest to remark that the geometric concept of abnormality and, indeed, constant abnormality plays an important role in the advance made by Marris [16,17] in his investigation of Ericksen's problem to determine all deformations that can be sustained by a perfectly elastic, isotropic, incompressible body subject only to surface tractions [18]. Marris' contribution to the study of Ericksen's problem has recently been discussed in a survey on universal solutions in elasticity by Saccomandi [19]. Universal states in anti-plane shear and connections with hydrodynamics have also been discussed by Knowles [20].

An analogue of Gilbarg's problem may also be formulated in the context of magnetohydrodynamics. In that case, it has been shown that the integrable Pohlmeyer–Lund–Regge model subject to a volume-preserving constraint arises as an exact reduction of the equilibrium equations [21]. Moreover, the above-mentioned Heisenberg spin connection is retrieved in the hydrodynamic or magnetohydrostatic limit.

2. The class of hydrodynamic motions

Here, we consider the classical system of steady hydrodynamics

$$\operatorname{div} \boldsymbol{q} = 0, \qquad \rho(\boldsymbol{q} \cdot \nabla)\boldsymbol{q} + \nabla p = \mathbf{0}, \tag{2.1}$$

where $\boldsymbol{q}$ is the fluid velocity and p, ρ are the pressure and constant density, respectively. In this context, the Gilbarg problem [1] may be formulated as follows:

Under what circumstances is a flow uniquely determined by its streamline pattern?

In [2], Prim established that "any flow is unique unless it has a constant velocity magnitude along each individual streamline." Thus, up to a trivial scaling of the velocity magnitude