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On the geometry and homology of certain simple stratified varieties

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Abstract

We study certain mild degenerations of algebraic varieties which appear in the analysis of a large class of supersymmetric theories, including superstring theory. We analyze Witten's σ-model [Nucl. Phys. B 403 (1993) 159] and find that the non-transversality of the superpotential induces additional singularities and a stratification of the ground state variety. This stratified variety admits certain homology groups such that $\oplus_q H^{2q}$ satisfies the "Kähler package" of requirements [Ann. Math. Studies 102 (1982) 303]. Also, this $\oplus_q H^{2q}$ extends the "flopped" pair of small resolutions [Nucl. Phys. B 416 (1994) 414; Nucl. Phys. B 330 (1990) 49; Commun. Math. Phys. 119 (1988) 431] to an "(exo)flopped" triple, and is compatible with both mirror symmetry [S.-T. Yau (Ed.), Mirror Manifolds, International Press, Hong Kong, 1990; B. Greene, S.-T. Yau (Eds.), Mirror Manifolds II, International Press, Hong Kong, 1996] and string theory [Mod. Phys. Lett. A 12 (1997) 521; Nucl. Phys. B 451 (1995) 96] results. Finally, we revisit the conifold transition [Nucl. Phys. B 330 (1990) 49] as it applies in our formalism.

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1. Introduction, results and summary

In string theory, rather than being an assumed arena, the spacetime is identified with the dynamically determined 'ground state variety' of a (supersymmetric) σ-model [11,20,23].

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In the simplest physically interesting and non-trivial case [3,23], the spacetime is of the form $M^{3,1} \times K$, where K is a compact Calabi-Yau three-fold modeled from the (bosonic subset of the) 'field space' of the σ-model[1], $\mathcal{F} = \{p, s_0, \ldots, s_4\} \simeq \mathbb{C}^6$, which admits a $\mathbb{C}^*$ action:

$$\hat{\lambda} : \{p, s_0, \ldots, s_4\} \mapsto \{\lambda^{-5} p, \lambda s_0, \ldots, \lambda s_5\}, \quad \lambda \in \mathbb{C}^*. \tag{1}$$

The 'ground state variety' is defined to be [2,23]

$$\mathcal{V} \stackrel{\text{def}}{=} [(\partial W)^{-1}(0) - 0]/\hat{\lambda}, \tag{2}$$

with the $\hat{\lambda}$-invariant holomorphic 'superpotential'

$$W \stackrel{\text{def}}{=} p \cdot G(s). \tag{3}$$

Alternatively, we denote by $\hat{\hat{\lambda}}$ the $|\lambda| = 1$ restriction of the map (1), and define the 'potential'

$$U_r \stackrel{\text{def}}{=} \|\partial W\|^2 + D_r^2, \tag{4}$$

where

$$D_r \stackrel{\text{def}}{=} \|s\|^2 - 5|p|^2 - r, \quad r \in \mathbb{R}. \tag{5}$$

Then

$$\mathcal{V} \simeq [U_r^{-1}(0) - 0]/\hat{\hat{\lambda}}. \tag{6}$$

Due to the positive definiteness of U_r,

$$U_r^{-1}(0) = (\partial W)^{-1}(0) \cap D_r^{-1}(0). \tag{7}$$

Furthermore, the $\hat{\lambda}$-invariance of $W = pG$ implies that $G(s)$ is a degree-5 homogeneous complex polynomial

$$G(\lambda s_0, \ldots, \lambda s_4) = \lambda^5 G(s_0, \ldots, s_4), \tag{8}$$

whereupon the zero locus of ∂W is the intersection of the cones

$$(\partial W)^{-1}(0) = G^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0). \tag{9}$$

The above definition may then be rephrased as follows.

Definition 1. Given the polynomials $G(s)$ and D_r as defined in Eqs. (8) and (5), respectively, the 'ground state variety' is

$$\begin{aligned} \mathcal{V} &= \{G^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0) - 0\}/\hat{\lambda} \\ &= \{G^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0) \cap D_r^{-1}(0) - 0\}/\hat{\hat{\lambda}}, \end{aligned} \tag{10}$$

where the S^1-action, $\hat{\hat{\lambda}}$, in the latter (symplectic) quotient is the $|\lambda| = 1$ restriction of the $\mathbb{C}^*$-action (1) in the former (holomorphic) quotient.

$\mathcal{V}^+$ ($\mathcal{V}^-$) shall denote the restriction of $\mathcal{V}$ to positive (negative) values of r in Eq. (5).

[1] To avoid obscuringly complicated notation, we focus on a simple example and discuss generalizations later.