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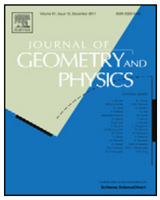

Affine holomorphic quantization

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ABSTRACT

We present a rigorous and functorial quantization scheme for affine field theories, i.e., field theories where local spaces of solutions are affine spaces. The target framework for the quantization is the general boundary formulation, allowing to implement manifest locality without the necessity for metric or causal background structures. The quantization combines the holomorphic version of geometric quantization for state spaces with the Feynman path integral quantization for amplitudes. We also develop an adapted notion of coherent states, discuss vacuum states, and consider observables and their Berezin–Toeplitz quantization. Moreover, we derive a factorization identity for the amplitude in the special case of a linear field theory modified by a source-like term and comment on its use as a generating functional for a generalized S-matrix.

1. Introduction

Ever since its inception, efforts have been made to put quantum field theory on an axiomatic basis. There are multiple objectives behind such undertakings. Conceptually, one would like to have a better understanding of what quantum field theory "really is" (and what it is not), possibly including an elucidation of aspects of the meaning or interpretation of quantum theory itself. Mathematically, an axiomatic system offers a rigorous definition and a context to make mathematically precise statements about certain quantum field theories or quantum field theory as such. Finally, an axiomatic formulation may help to indicate how quantum field theories can be extended to realms where they have not previously been experimentally tested. An important example for the latter is the extension from Minkowski space to more general curved spacetime.

An axiomatic approach that has proven particularly useful in this latter respect is *algebraic quantum field theory* (AQFT) [1]. In AQFT the causal structure of spacetime is intimately entwined with the algebraic structure of the objects of the quantum theory. This has advantages and disadvantages. Most notably, this leads to a very concise way of encoding local physics in a spacetime region, with just one core mathematical structure (a von Neumann or C^* algebra) per spacetime region. Moreover, in quantization prescriptions this structure is directly linked to the classical observables in that spacetime region. This conciseness combined with mathematical rigor has justifiably fascinated physicists and mathematicians over the decades, making it today the best developed axiomatic approach to quantum field theory.

On the other hand, the central role played by causality in the core structure of AQFT makes it indispensable as a fixed ingredient of spacetime. This precludes the direct applicability of AQFT to situations where such a structure is not a priori given.

This limitation, which is even more stringent in most other approaches to quantum field theory, has motivated a new axiomatic approach, called the *general boundary formulation* (GBF). The GBF has been put forward with the express aim of disentangling the elementary mathematical objects of a theory (in this case states, amplitudes, and observables) and

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their basic physical interpretation, from the metric or causal structure of spacetime. This is achieved on the one hand by explicitly localizing states on hypersurfaces and amplitudes in spacetime regions [2] in the spirit of topological quantum field theory [3]. On the other hand this requires an extension of the probability postulates of quantum theory for amplitudes [4] and observables [5]. While still considerably less developed than, say, AQFT, the GBF offers the perspective of further extending the realm of quantum field theory to contexts where spacetime is not equipped with a predetermined metric or causal background structure. It is widely expected that a quantum theory of gravity should live precisely in such a "background independent" context.

Most realistic quantum field theories are obtained or at least motivated through a process of quantization starting with a classical field theory. It is thus important for the usefulness of a given axiomatic approach that there be quantization prescriptions that produce the elementary objects which are the subject of the axioms starting from data encoding a classical field theory. In the case of the GBF the quantization prescription most straightforwardly adapted from well known tools of quantum (field) theory is Schrödinger–Feynman quantization [4,6], which combines the Schrödinger representation [7] for state spaces with the Feynman path integral [8] for amplitudes. This quantization prescription has been successfully applied in various contexts including a non-perturbative integrable model [9], a generalization of the perturbative S-matrix [10], and in curved spacetime [11,12]. Even though many of these applications lead to structures that rigorously satisfy the axioms, the quantization prescription itself is not rigorously formulated, at least not in its present form.

Ideally, quantization should not only be rigorous, but should provide something like a functor from a category of classical theories to a category of quantum theories. For the GBF such a functorial quantization scheme has indeed been described recently for the case of linear field theory [13]. There, the concept of a linear classical field theory is axiomatized and a construction is given that produces from the elementary objects of such a classical theory the elementary objects of a quantum field theory in the framework of the GBF. In particular, it is proven that the objects of the quantum theory obtained in this way do indeed satisfy the axioms of the GBF. Moreover, although it is not made explicit there, this construction is functorial, and in many ways so. For example, for a given system of spacetime hypersurfaces and regions we obtain a functor if we take the categories of classical and quantum field theories with morphisms given by the respective notion of "subtheory": On the classical side a "subtheory" is obtained by restricting the local spaces of solutions consistently to subspaces, while on the quantum side a "subtheory" is obtained by decomposing the local Hilbert spaces of states into tensor products and selecting one component in a consistent way. Other possibilities for choices of categories include ones where each object carries its own system of hypersurfaces and regions etc.

A classical linear field theory is formalized in [13] as follows: For each region in spacetime we are given a real vector space of solutions of the field equations. Also, for each hypersurface in spacetime we are given a real vector space of germs of solutions. The latter spaces are moreover equipped with non-degenerate symplectic forms. Then, the natural maps from the former spaces to the latter (restricting solutions in regions to neighborhoods of the boundary) have to yield Lagrangian subspaces with respect to these symplectic forms. Although perhaps not obviously so, these conditions are well motivated from Lagrangian field theory. An additional ingredient which might be seen as structure already pertaining to the quantum realm is a compatible complex structure on the solution space for each hypersurface. This summarizes the axioms given in [13] for a classical linear field theory in an informal language.

The quantization prescription consists then of a combination of a version of geometric quantization for hypersurfaces and a certain integral quantization for regions. For each hypersurface, the construction of the associated Hilbert space of states is equivalent to the usual Fock space construction, where the phase space (here really the space of germs of classical solutions in a neighborhood of the hypersurface) with additional symplectic and complex structure is seen as the (dual of the) 1-particle Hilbert space. However, it is realized concretely as a space of holomorphic functions in the spirit of Bargmann. From the point of view of geometric quantization this is really the space of Kähler polarized sections of the prequantum bundle. For each region, the quantization prescription in [13] is given by a seemingly ad hoc integral prescription, although verified by providing the "right" results in certain examples.

In the present paper we consider *affine field theory*, as a first case of a rigorous and functorial quantization prescription targeting the GBF beyond linear field theory. By affine field theory we mean here field theory with affine spaces of local solutions and such that the natural symplectic forms associated to hypersurfaces are invariant with respect to the affine structure in addition to being non-degenerate. In many ways this can be seen as a generalization of the linear case and its treatment in [13]. For hypersurfaces, this requires a refinement of the geometric quantization prescription (Section 2.3), clarifying the role of the prequantum bundle and its relevant trivializations. For regions, we motivate the quantization as a variant of the Feynman path integral prescription (Section 2.4), thus justifying at the same time the origin of the prescription given in [13] as a special case of this.

Based on a suitable geometric setting for spacetime (Section 3.1), the axioms for classical field theory (Section 3.2) are a relatively straightforward generalization of those for linear field theory given in [13]. However, they involve additional structural elements from Lagrangian field theory (see Sections 2.1 and 2.2), notably the action and the symplectic potential. Also, they are considerably more extensive as both local spaces of solutions and their tangent spaces need to be kept track of separately since they are no longer canonically identified.

The central part of this paper is Section 4 where the quantization prescription is specified rigorously and the validity of the GBF core axioms (listed in Section 3.3) is proven. As in [13] the Hilbert spaces of states associated to hypersurfaces are realized concretely as spaces of functions (Section 4.1). However, the domain spaces (or rather their extensions) for these functions do not directly carry measures as in [13]. Rather, any choice of base point gives rise to an identification with a