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One-harmonic invariant vector fields on three-dimensional Lie groups

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ABSTRACT

We determine all left-invariant vector fields on three-dimensional Lie groups which define harmonic sections of the corresponding tangent bundles, equipped with the complete lift metric.

1. Introduction

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. A vector field X determines a section of the tangent bundle and therefore it is natural to investigate vector fields from the point of view of the corresponding maps $X : M \to TM$. Harmonic and minimal vector fields have been investigated in the literature by considering the Sasaki lift of g as the induced metric on TM (see [1–3] and the references therein). However, when investigating harmonicity problems it is also relevant to endow TM with the complete lift metric g^c, which is of neutral signature (n, n) [4–7].

Vector fields on pseudo-Riemannian manifolds (M, g) defining harmonic sections $X : (M, g) \to (TM, g^c)$ have been investigated in the literature under different names like geodesic vector fields [8], infinitesimal harmonic transformations [9,10] and 1-harmonic vector fields [11] since the harmonicity property is equivalent to the vanishing of the linear part of the tension field of the local one-parameter group of infinitesimal point transformations, i.e., trace $\mathcal{L}_X \nabla = 0$, where ∇ is the Levi-Civita connection of (M, g) and $\mathcal{L}$ stands for the Lie derivative. This property has been recently linked with the existence of Ricci solitons in [12], by showing that any Ricci soliton is a 1-harmonic vector field.

The purpose of this paper is to determine all left-invariant 1-harmonic vector fields on three-dimensional Lie groups. Obviously affine Killing vector fields (and hence Killing vector fields) are 1-harmonic, so we emphasize the existence of the non-affine Killing ones.

We organize this paper as follows. In Section 2 we review the description of all three-dimensional Lorentzian Lie algebras and some basic facts on 1-harmonic vector fields. We analyze the existence of left-invariant Killing, affine Killing and 1-harmonic vector fields on unimodular Lorentzian Lie groups in Section 3, while the non-unimodular Lorentzian case is considered in Section 4. In each case we determine the corresponding vector subspaces of Killing, affine Killing and 1-harmonic left-invariant vector fields. Finally, in Section 5 the Riemannian case is analyzed; we will omit the details since

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the results are obtained essentially as in Sections 3 and 4. Moreover, in Section 5 it is also shown that the class of left-invariant 1-harmonic vector fields is strictly larger than the class of Lorentzian Ricci solitons.

2. Preliminaries

2.1. Three-dimensional Lorentzian Lie algebras

Let $\times$ denote the Lorentzian vector product on $\mathbb{R}^3_1$ induced by the product of the para-quaternions (i.e., $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature $(++-)$). Then $[Z, Y] = L(Z \times Y)$ defines a Lie algebra, which is unimodular if and only if L is a self-adjoint endomorphism of $\mathfrak{g}$ [13,14]. Considering the different Jordan normal forms of L, we have the following four classes of unimodular three-dimensional Lorentzian Lie algebras.

Type Ia. If L is diagonalizable with eigenvalues $\{\alpha, \beta, \gamma\}$ with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, then the corresponding Lie algebra is given by

$$(\mathfrak{g}_{\mathrm{Ia}}) \quad [e_1, e_2] = -\gamma e_3, \qquad [e_1, e_3] = -\beta e_2, \qquad [e_2, e_3] = \alpha e_1.$$

Type Ib. Assume L has a complex eigenvalue. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \gamma & -\beta \\ 0 & \beta & \gamma \end{pmatrix}, \quad \beta \neq 0$$

and thus the corresponding Lie algebra is given by

$$(\mathfrak{g}_{\mathrm{Ib}}) \quad [e_1, e_2] = \beta e_2 - \gamma e_3, \qquad [e_1, e_3] = -\gamma e_2 - \beta e_3, \qquad [e_2, e_3] = \alpha e_1.$$

Type II. Assume L has a double root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{2} + \beta & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} + \beta \end{pmatrix}$$

and thus the corresponding Lie algebra is given by

$$(\mathfrak{g}_{\mathrm{II}}) \quad [e_1, e_2] = \frac{1}{2} e_2 - \left(\beta - \frac{1}{2}\right) e_3, \qquad [e_1, e_3] = -\left(\beta + \frac{1}{2}\right) e_2 - \frac{1}{2} e_3, \qquad [e_2, e_3] = \alpha e_1.$$

Type III. Assume L has a triple root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \alpha & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \alpha \end{pmatrix}$$

and thus the corresponding Lie algebra is given by

$$(\mathfrak{g}_{\mathrm{III}}) \quad [e_1, e_2] = -\frac{1}{\sqrt{2}} e_1 - \alpha e_3, \qquad [e_1, e_3] = -\frac{1}{\sqrt{2}} e_1 - \alpha e_2, \qquad [e_2, e_3] = \alpha e_1 + \frac{1}{\sqrt{2}} (e_2 - e_3).$$

Next we treat the non-unimodular case. First of all, recall that a solvable Lie algebra $\mathfrak{g}$ belongs to the special class $\mathfrak{S}$ if $[x, y]$ is a linear combination of x and y for any pair of elements in $\mathfrak{g}$. Any left-invariant metric on $\mathfrak{S}$ is of constant sectional curvature [15,16]. Now, consider the unimodular kernel $\mathfrak{u} = \ker(\mathrm{trace}\ ad : \mathfrak{g} \to \mathbb{R})$. It follows from [17] that non-unimodular Lorentzian Lie algebras of non-constant sectional curvature are given, with respect to a suitable basis $\{e_1, e_2, e_3\}$, by

$$(\mathfrak{g}_{\mathrm{IV}}) \quad [e_1, e_2] = 0, \qquad [e_1, e_3] = \alpha e_1 + \beta e_2, \qquad [e_2, e_3] = \gamma e_1 + \delta e_2, \qquad \alpha + \delta \neq 0,$$

where one of the following holds:

IV.1 $\{e_1, e_2, e_3\}$ is orthonormal with $g(e_1, e_1) = -g(e_2, e_2) = -g(e_3, e_3) = -1$ and the structure constants satisfy $\alpha\gamma - \beta\delta = 0$.