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Nonobservable space dimensions and the discreteness of time ☆

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Abstract

We present a simple dynamical system model for the effect of nonobservable space dimensions on the observable ones. There are three premises. A: Orbits consist of flows of probabilities [Ilya P. The end of certainty. NY: The Free Press; 1996] (which is the case in the setting of quantum mechanics). B: The orbits of probabilities are induced by (continuous time) differential or partial differential equations. C: The observable orbit is a flow of marginal probabilities where the nonobservable space dimensions are averaged out. A theorem is presented which proves that under certain general conditions the transfer of marginal probabilities cannot be achieved by continuous time dynamical systems acting on the space of observable variables but can be achieved by discrete time dynamical systems.

1. Introduction

The realm of general relativity (GR) consists of massive structures and great distances while the realm of quantum mechanics (QM) consists of the structures such as photons and quarks on tiny scales. In most circumstances one or the other theory applies without conflict. However, in extreme situations, such as black holes, both theories are needed for accurate theoretical analysis. As subatomic particles possess mass, and space possesses structure on all scales, it has been a great challenge for many decades to apply GR on very small scales where the smooth spatial structures of large scales give way to precipitous spatial landscapes. The inability to predict dynamics on these tiny scales is captured by the Heisenberg Principle which asserts that point orbits are meaningless, that the only meaningful dynamic trajectories on these scales are those of probabilities. As the scales become larger, these probability orbits are supported on ever narrower segments of space–time, thereby delineating accurately trajectories of points. As the objective of this note is to present an idea toward the melding of QM and GR we shall deal exclusively with orbits of probabilities.

The assumption of space and time continuity together with the machinery of calculus that depends on continuity are the pillars on which the theories of QM and GR rest. The objective of this note is to provide evidence for the untethering of time from the constraint of continuity. The motivation for this is based on the fact that the constraint of time continuity is so great that only very restricted dynamical behavior is possible. Our approach is based on ideas from the

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modern theory of nonlinear dynamics (chaos) which shows that the range of dynamical behavior for even the simplest discrete time systems are incredibly rich and, as we shall argue, rich enough to accommodate the behavior of particles in extreme situations where the effects of gravitational attraction on tiny particles must be taken into account. Many researchers presented various approaches to the problem of discreteness of space–time. We give only a few references and direct the readers to further literature therein [2–4,6–10].

The gap between time points we have in mind is of the order of the Planck scale (10^{-34} s) and as such it is impossible with the present state of technology to perform experiments that might establish the discreteness of time. Hence, our evidence is of a mathematical nature.

In order to capture the larger range of dynamical behavior needed in extreme situations, string theory purportedly resolves the incompatibility problem by modifying the equations of GR on small scales. But there is a price for this accomplishment in that—to account accurately for quantum effects—space is attributed to have, not three, but nine dimensions. Three dimensions are observable, while the other six dimensions are curled up in tiny, essentially nonobservable strings. The 'extra' dimensions provide the additional freedom needed to model the dynamics of a unified QM and GR theory. It is interesting that, in the course of effecting the unification of QM and GR, string theory itself points in the direction of discrete space and time as do other theories such as black hole thermodynamics and loop quantum gravity [11].

It is our objective to suggest that invoking discrete time at the very outset can yield new and interesting insights. In this note we present a simple dynamical system model for the effect of nonobservable space dimensions on the observable ones. There are three premises on which our model is based:

A. Orbits consist of flows of probabilities [12] which is the case in QM.
B. The orbits of probabilities are induced by (continuous time) dynamical systems.
C. An observable orbit is a flow of marginal probabilities where the unobservable space dimensions are averaged out.

Note that point orbits can be equivalently represented as point measure orbits. Thus, the point orbits of GR theory can be viewed in the unified framework of probability orbits.

In Section 2 we present the framework and notation for this note. In Section 3 we state our main result, which suggests a possible relationship between nonobservable space dimensions and the discreteness of time. In Section 4 we present a number of examples.

2. Framework and notation

It is now common knowledge that even simple one-dimensional maps have the ability to describe very complicated dynamical behavior of biological and mechanical systems. Modeling dynamics by a map offers much more variety of behavior than do differential equations whose solutions are greatly restricted by time continuity and cannot exhibit chaotic behavior in low dimensions. Describing dynamical behavior by iterating a map, which can arise as a Poincare section or by direct modeling offers many benefits from an analysis perspective. Once a map is determined, the long term statistical behavior is described by a probability density function, which can be obtained by measurement of the system or by mathematical means using the Frobenius–Perron operator [1] as follows: let I denote the state space of a dynamical system and let $\tau : I \to I$, describe the dynamics of the system. The dynamics is described by a probability density function f associated with the unique (absolutely continuous invariant) measure μ. The invariance of the density f is stated mathematically by the following equation:

$$\int_A f\,\mathrm{d}x = \int_{\tau^{-1}(A)} f\,\mathrm{d}x$$

for any (measurable) set $A \subset \mathbb{R}$. The Frobenius–Perron operator, $P_\tau f$, acts on the space of integrable functions and is defined by

$$\int_{\tau^{-1}(A)} f\,\mathrm{d}x = \int_A P_\tau f\,\mathrm{d}x.$$

The operator P_τ transforms a probability density function (pdf) into a pdf under the transformation τ. If τ is piecewise smooth and piecewise differentiable on a partition of n subintervals, we have the following representation for P_τ [1, Chapter 4]: