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A new Jacobi elliptic function rational expansion method and its application to (1 + 1)-dimensional dispersive long wave equation

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Abstract

With the aid of computerized symbolic computation, a new elliptic function rational expansion method is presented by means of a new general ansätz and is very powerful to uniformly construct more new exact doubly-periodic solutions in terms of rational formal Jacobi elliptic function of nonlinear evolution equations (NLEEs). As an application of the method, we choose a (1 + 1)-dimensional dispersive long wave equation to illustrate the method. As a result, we can successfully obtain the solutions found by most existing Jacobi elliptic function methods and find other new and more general solutions at the same time. Of course, more shock wave solutions or solitary wave solutions can be gotten at their limit condition.

1. Introduction

In recent years, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. With the development of soliton theory, There has been a great amount of activities aiming to find methods for exact solution of nonlinear differential equations, such as Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, varied tanh methods, varied Jacobi elliptic function methods, variable separation approach, Painlevé method, homogeneous balance method, similarity reduction method and so on [1–12].

Among those, the direct ansätz method [6–12] provides a straightforward and effective algorithm to obtain such particular solutions for a large number of nonlinear equations, in which the starting point is the ansätz that the solution sought is expressible as a finite series of special function, such as tanh function, sech function, tan function, sec function, sine–cosine function, Weierstrass elliptic function, theta function and Jacobian elliptic function [13,14].

In this paper a new Jacobi elliptic function rational expansion method is presented by means of a new general ansätz and is more powerful than above exiting Jacobi elliptic function methods [10–12] to uniformly construct more new exact

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doubly-periodic solutions in terms of rational formal elliptic function of nonlinear evolution equations (NLEEs). The algorithm and its applications are demonstrated later.

This paper is organized as follows. In Section 2, we summarize the elliptic function rational expansion method. In Section 3, we apply the generalized method to (1 + 1)-dimensional dispersive long wave equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the Jacobi elliptic function rational expansion method

In the following we would like to outline the main steps of our general method:

Step 1. For a given nonlinear partial differential equation(NPDE) system with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x,y,t) = u_i(\xi), \quad \xi = x + ly - \lambda t, \tag{2.2}$$

where l and λ are constants to be determined later. Then the nonlinear partial differential Eq. (2.1) is reduced to a nonlinear ordinary differential equation(ODE):

$$G_i(u_i, u_i', u_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal elliptic function expansion in the following forms:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left(a_{ij} \frac{\mathrm{sn}^j(\xi)}{(\mu \mathrm{sn}(\xi) + 1)^j} + b_{ij} \frac{\mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu \mathrm{sn}(\xi) + 1)^j} \right). \tag{2.4}$$

Notice that

$$\begin{aligned} \frac{\mathrm{d}u_i}{\mathrm{d}\xi} = & \sum_{j=1}^{m_i} \frac{\mathrm{dn}(\xi)(a_{ij} j(\mathrm{sn}(\xi))^{j-1}\mathrm{cn}(\xi) - b_{ij}(\mathrm{sn}(\xi))^{j-2} - b_{ij}\mu(\mathrm{sn}(\xi))^{j-1})}{(\mu \mathrm{sn}(\xi) + 1)^{j+1}} \\ & + \sum_{j=1}^{m_i} \frac{\mathrm{dn}(\xi)(b_{ij} j(\mathrm{sn}(\xi))^{j-2} - b_{ij} j(\mathrm{sn}(\xi))^{j})}{(\mu \mathrm{sn}(\xi) + 1)^{j+1}}, \end{aligned} \tag{2.5}$$

where $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, $\mathrm{dn}\,\xi$, $\mathrm{ns}\,\xi$, $\mathrm{cs}\xi$, and $\mathrm{ds}\,\xi$ etc. are Jacobi elliptic functions, which are double periodic and posses the following properties:

1. Properties of triangular function

$$\mathrm{cn}^2\xi + \mathrm{sn}^2\xi = \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1, \tag{2.6.1}$$

$$\mathrm{ns}^2\xi = 1 + \mathrm{cs}^2\xi, \quad \mathrm{ns}^2\xi = \mathrm{m}^2 + \mathrm{ds}^2\xi, \tag{2.6.2}$$

2. Derivatives of the Jacobi elliptic functions

$$\mathrm{sn}'\xi = \mathrm{cn}\,\xi\mathrm{dn}\,\xi, \quad \mathrm{cn}'\xi = -\mathrm{sn}\,\xi\mathrm{dn}\,\xi, \quad \mathrm{dn}'\xi = -m^2\mathrm{sn}\xi\mathrm{cn}\,\xi, \tag{2.7.1}$$

$$\mathrm{ns}'\xi = -\mathrm{ds}\,\xi\mathrm{cs}\,\xi, \quad \mathrm{ds}'\xi = -\mathrm{cs}\,\xi\mathrm{ns}\,\xi, \quad \mathrm{cs}'\xi = -\mathrm{ns}\xi\mathrm{ds}\,\xi, \tag{2.7.2}$$

where m is a modulus. The Jacobi–Glaisher functions for elliptic function can be found in Refs. [13,14].

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly-periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[u_i^{(\alpha)}] = n_i + \alpha, \quad D[u_i^{\beta}(u_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s. \tag{2.8}$$