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Estimating the bounds for the Lorenz family of chaotic systems ☆

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Abstract

In this paper, we derive a sharper upper bound for the Lorenz system, for all the positive values of its parameters a, b and c. Comparing with the best result existing in the current literature, we fill the gap of the estimate for $0 < b \leqslant 1$ and get rid of the singularity problem as $b \to 1^+$. Furthermore, for $a > 1$, $1 \leqslant b < 2$, we obtain a more precise estimate. Along the same line, we also provide estimates of bounds for a unified chaotic system for $0 \leqslant \alpha < \frac{1}{29}$. When $\alpha = 0$, the estimate agrees precisely with the known result. Finally, the two-dimensional bounds with respect to $x - z$ for the Chen system, Lü system and the unified system are established.

1. Introduction

In 1963, Lorenz found the first chaotic system, which is a third-order autonomous system with only two multiplication-type quadratic terms but displays very complex dynamical behaviors [1]. In 1999, Chen found another similar but topologically non-equivalent chaotic system—the Chen system [2], which is a dual system to the Lorenz system in the sense of a canonical form introduced by Vaněček and Čelikovský [3]: After separating the system into linear and quadratic parts, in the linear part of the system described by the matrix $A = [a_{ij}]_{3\times 3}$, the Lorenz system satisfies the condition $a_{12}a_{21} > 0$ while the Chen system satisfies $a_{12}a_{21} < 0$. In 2002, Lü et al. found another chaotic system, the Lü system [4], which satisfies $a_{12}a_{21} = 0$. Very recently, Lü et al. introduced a unified chaotic system [5] which describes a large family of chaotic systems containing the Lorenz and Chen systems as two extremes and the Lü system as a transition in between. Recently, there are some analytical results reported about these chaotic systems, which are called the Lorenz family [6–10].

A chaotic system is bounded, and the estimate of its bound is important in chaos control, chaos synchronization, and their applications. Technically, this is also a very difficult task. In 1987, Leonov et al. investigated the boundedness of the Lorenz system [11], and Pogromsky et al. investigated the bound of the trajectories of the Lorenz system [12]. In 2003, Zhou et al. investigated the bound of the Chen system [13].

In this paper, we derive an sharper upper bound for the Lorenz system, for all the positive values of its parameters a, b and c. Comparing with the best result existing in the current literature [11], we fill the gap of the estimate for $0 < b \leqslant 1$ and get rid of the singularity problem as $b \to 1^+$. Furthermore, for $a > 1$, $1 \leqslant b < 2$, we obtain a more precise estimate. Along the same line, we also provide estimates of bounds for a unified chaotic system for $0 \leqslant \alpha < \frac{1}{29}$. When $\alpha = 0$, the

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estimate agrees precisely with the known result given in [11]. Finally, the two-dimensional bounds with respect to $x - z$ for the Chen system, Lü system and the unified system are established.

2. The estimate of the bound for the unified chaotic system

We first discuss the unified chaotic system [5].

Lemma 1. *Consider the ellipsoid*

$$\Gamma = \left\{ (x,y,z) \left| \; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1, \; a > 0, \; b > 0, \; c > 0 \right. \right\}.$$

Denote $G = x^2 + y^2 + z^2$, $H = x^2 + y^2 + (z - 2c)^2$, $(x,y,z) \in \Gamma$. *Then*

$$G1 \equiv \max_{(x,y,z)\in\Gamma} G = H1 \equiv \max_{(x,y,z)\in\Gamma} H = \begin{cases} \frac{a^4}{a^2-c^2}, & a \geqslant b, \; a \geqslant \sqrt{2}c, \\ \frac{b^4}{b^2-c^2}, & b > a, \; b \geqslant \sqrt{2}c, \\ 4c^2, & a < \sqrt{2}c, \; b < \sqrt{2}c. \end{cases} \tag{1}$$

Proof. Obviously, $\max_{(x,y,z)\in\Gamma} G = \max_{(x,y,z)\in\Gamma} H$. Define

$$F = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} - 1\right),$$

and let

$$\frac{1}{2}F'_x = x\left(1 + \frac{\lambda}{a^2}\right) = 0, \tag{2}$$

$$\frac{1}{2}F'_y = y\left(1 + \frac{\lambda}{b^2}\right) = 0, \tag{3}$$

$$\frac{1}{2}F'_z = z + \lambda\frac{z-c}{c^2} = 0. \tag{4}$$

(i) When $\lambda \neq -a^2$, $\lambda \neq -b^2$, we have $(x_0, y_0, z_0) = (0,0,0)$ or $(0,0,2c)$, and $G1 = 0$ or $G1 = 4c^2$ correspondingly.
(ii) When $\lambda = -a^2$ $(a \neq b)$ and $a \geqslant \sqrt{2}c$, Eqs. (2)–(4) have the following solution: $x_0 = \pm\frac{a^2}{a^2-c^2}\sqrt{a^2 - 2c^2}$, $y_0 = 0$, $z_0 = \frac{a^2c}{a^2-c^2}$, and $G1 = \frac{a^4}{a^2-c^2}$.
(iii) When $\lambda = -b^2$ $(a \neq b)$, if $b \geqslant \sqrt{2}c$, it follows from (2)–(4) that $x_0 = 0$, $y_0 = \pm\frac{b^2}{b^2-c^2}\sqrt{b^2 - 2c^2}$, $z_0 = \frac{b^2c}{b^2-c^2}$, and $G1 = \frac{b^4}{b^2-c^2}$.
(iv) When $\lambda = -a^2, (a = b)$, we get from (2)–(4) that $z_0 = \frac{a^2c}{a^2-c^2}$, and $G1 = \frac{a^4}{a^2-c^2}$.

Summarizing (i)–(iv) above, the proof of the lemma is completed. □

The unified chaotic system is described by [5]

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x), \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \dot{z} = xy - \frac{8+\alpha}{3}z. \end{cases} \tag{5}$$

This system is chaotic for all $0 \leqslant \alpha \leqslant 1$; with $\alpha = 0$ it reduces to the original Lorenz system and with $\alpha = 1$ it is the original Chen system.

Theorem 1. *When* $0 \leqslant \alpha < \frac{1}{29}$, *the unified chaotic system* (5) *is contained in the sphere*

$$\Omega = \{(x,y,z) | x^2 + y^2 + (z - 38 + 10\alpha)^2 = R^2\}, \tag{6}$$