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An approach for constructing loop algebra via exterior algebra and its applications

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Abstract

With the help of some properties of exterior algebra defined by us, a general approach for constructing multi-component matrix loop algebra is proposed. By making use of the approach, a new 3M loop algebra $\widetilde{X}$ is constructed. This algebra can be easily reduced to the existing multi-component loop algebra. Another an new extended loop algebra $\widetilde{Y}$ is also presented. As their applicable examples, a generalized multi-component AKNS hierarchy with arbitrary smooth functions and a generalized multi-component KN hierarchy are worked out. As a reduction cases of the first hierarchy, the standard multi-component heat-conduction equation and a coupled generalized multi-component Burgers equation are given. The approach presented in the paper can be used generally.

1. Introduction

Various efficient techniques have been proposed to obtain integrable hierarchies of soliton equations, such as the KdV system, the AKNS system, the KN system, the Schrödinger system, Burgers system and so on [1–11]. As for the multi-component integrable hierarchies, there have been developments in Refs. [12,13]. Ma and Zhou [14] have made use of the generalized Tu scheme to arrive at the multi-component AKNS hierarchy and other interesting results. A Lie algebra with 3M dimensions (here M denotes a positive integer)has been constructed in [15] so that a new multi-component integrable hierarchy was presented. By employing the method proposed in [15], some other multi-component integrable hierarchies can be generated. However, it is not very easy to construct the Lie algebra in [15]. The purpose of this paper is to adapt a general scheme to present a series of Lie algebras so that some multi-component integrable hierarchies of soliton equations can be easily obtained? Specially, by using the exterior algebra defined by us, any finite-dimensional Lie algebra can be obtained. As an applicable illustrative example, a 3M dimensional Lie algebra $\widetilde{X}$ is constructed. The corresponding loop algebra is also produced. It follows that an isospectral problem is established. By taking advantage of the Tu scheme, a generalized multi-component hierarchy with arbitrary derivative multi-functions is given. As its degenerative cases, the well-known multi-component heat-conduction equation and a coupled generalized multi-component Burgers equation are obtained. By constructing a 4M dimensional extended loop algebra $\widetilde{Y}$ of the loop algebra $\widetilde{X}$, another isospectral problem is given. As an application of the loop algebra $\widetilde{Y}$, a generalized

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multi-component KN hierarchy is worked out. Through the exterior algebra method, we can construct many multi-component matrix loop algebras, showing that the approach presented in this paper can be used in general.

2. A new type of Lie algebra

We begin with a few notations.

Definition 1. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_M)^{\mathrm{T}}$, $\beta = (\beta_1, \beta_2, \dots, \beta_M)^{\mathrm{T}}$ be two column vectors, define their vector product $\alpha * \beta$ by

$$\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_M\beta_M)^{\mathrm{T}}. \tag{1}$$

By introducing the diagonal matrix $\widetilde{\alpha} = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_M)$,we have

$$\alpha * \beta = \widetilde{\alpha}\beta. \tag{2}$$

Definition 2. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_M)^{\mathrm{T}}$, $A = (0, \dots, 0, a_i, 0, \dots, 0)_{M\times N}$, we define their product by

$$\alpha \cdot A = A \cdot \alpha = (0, \dots, 0, \alpha * a_i, 0, \dots, 0),$$

where

$$a_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{iM} \end{pmatrix}, \quad a_{ij} \in \mathbf{R}$$

or $\mathbf{C}$ are real or complex numbers and are unrelated to the independent variables x, t.

Definition 3. Let $x^1 = (x_1^1, 0, 0), x^2 = (0, x_2^2, 0), x^3 = (0, 0, x_3^3)$ be linear independent real or complex $M \times 3$ matrices and are independent of x and t. Let X denote a linear space generated by $\{x^1, x^2, x^3\}$ i.e.

$$X = \{\omega_a | \omega_a = a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3\}. \tag{3}$$

$$a_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{iM} \end{pmatrix} \quad (i = 1, 2, 3)$$

stand for real or complex vectors,

$$x_i^i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iM} \end{pmatrix}, \quad x_{ij}^i \in \mathbf{R} \text{ or } \mathbf{C}.$$

Define a commutation operation $[\omega_a, \omega_b]$ in X by

$$[\omega_a, \omega_b] \equiv \omega_a \wedge \omega_b = 2[(a_1 * b_2 - a_2 * b_1) \cdot x^1 \wedge x^2 + (a_1 * b_3 - a_3 * b_1) \cdot x^1 \wedge x^3 + (a_2 * b_3 - a_3 * b_2) \cdot x^2 \wedge x^3], \tag{4}$$

$$x^1 \wedge x^2 = -x^2 \wedge x^1 = 2x^2, \quad x^1 \wedge x^3 = -x^3 \wedge x^1 = -2x^3, \quad x^2 \wedge x^3 = -x^3 \wedge x^2 = x^1, \quad x^i \wedge x^i = 0, \quad i = 1, 2, 3, \tag{5}$$

where the symbol $\wedge$ is called as an exterior product by us, $\omega_a, \omega_b \in X$. Then X along with (4) and (5) becomes a Lie algebra.

Specially, taking

$$x^1 = \tfrac{1}{2}(I_M, 0, 0), \quad x^2 = \tfrac{1}{2}(0, I_M, 0), \quad x^3 = \tfrac{1}{2}(0, 0, I_M), \tag{6}$$