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Transition to spatiotemporal chaos via stationary branching shocks and holes

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ABSTRACT

Spatiotemporal chaos in the complex Ginzburg–Landau equation is known to be associated with a rapid increase in the density of defects, which are isolated points at which the solution amplitude is zero and the phase is undefined. Recently there have been significant advances in understanding the details and interactions of defects and other coherent structures, and in the theory of convective and absolute stability. In this paper, the authors exploit both of these advances to update and clarify the onset of spatiotemporal chaos in the particular case of the complex Ginzburg–Landau equation with zero linear dispersion. They show that very slow increases in the coefficient of nonlinear dispersion cause a shock–hole (defect) pair to develop in the midst of a uniform expanse of plane wave. This is followed by a cascade of splittings of holes into shock–hole–shock triplets, culminating in spatiotemporal chaos at a parameter value that matches the change in absolute stability of the plane wave. The authors demonstrate a close correspondence between the splitting events and theoretical predictions, based on the theory of absolute stability. They also use measures based on power spectra and spatial correlations to show that when the plane wave is convectively unstable, chaos is restricted to localised regions, whereas it is extensive when the plane wave is absolutely unstable.

1. Introduction

Many spatially extended physical systems exhibit chaotic dynamics. One model in which such spatiotemporal chaos has been studied in detail is the (cubic) complex Ginzburg–Landau equation (CGLE), which arises as the amplitude equation near a standard supercritical Hopf bifurcation, and which has been applied effectively to a wide range of physical, chemical and biological systems [1,2]. In one space dimension, two different regimes of spatiotemporal chaos occur in the CGLE. In "phase chaos" the solution amplitude is bounded away from zero, so that there is long-range phase coherence, and the phase difference across the whole domain is constant [2–5]. By contrast, "defect chaos" is characterised by large oscillations in amplitude, including isolated points ("defects") at which the amplitude is zero. At such points the phase is not defined, destroying conservation of the overall phase difference [6–8]. Changes in parameter values from the phase chaos to defect chaos regimes are characterised by a rapid increase in the density of defects [7,9,10]; in some cases there is an overlap region ("bichaos") in which there is hysteresis in the defect density.

The last few years have seen two major advances that are relevant to these considerations. Firstly there has been significant progress in understanding of the details and interaction of coherent structures, including defects, in the CGLE and other spatially extended systems [11–14]. Secondly, the theory of convective and absolute stability has been placed on a firm and more accessible footing [15–19]. In this paper we exploit both of these advances to update and clarify the onset of spatiotemporal chaos in the particular case of the CGLE with zero linear dispersion.

The equation that we study is

$$\partial A/\partial t = \partial^2 A/\partial x^2 + A - (1+ic)|A|^2 A \tag{1}$$

where the complex field A is a function of space x and time t, and $c > 0$ is the real valued control parameter. Plane waves are a fundamental solution form for (1), with the general form $A = \sqrt{1-Q^2}e^{iQx-i\omega t}$, where $\omega = c(1-Q^2)$ and $-1 < Q < 1$.

In this study we investigated the dynamics emerging in simulations of (1) under the separated boundary conditions

$$A = 0 \quad \text{at } x = 0, \qquad \partial_x A = 0 \quad \text{at } x = L, \tag{2}$$

for a suitably large domain length L and random initial conditions other than at the boundaries (detailed below). These conditions have been used in the past to investigate the generation of plane waves in real systems such as oscillatory chemical reactions and ecological systems [20–23]. Under these boundary conditions, when $c < 1.110$, perturbations to $A \equiv 0$ evolve to a solution

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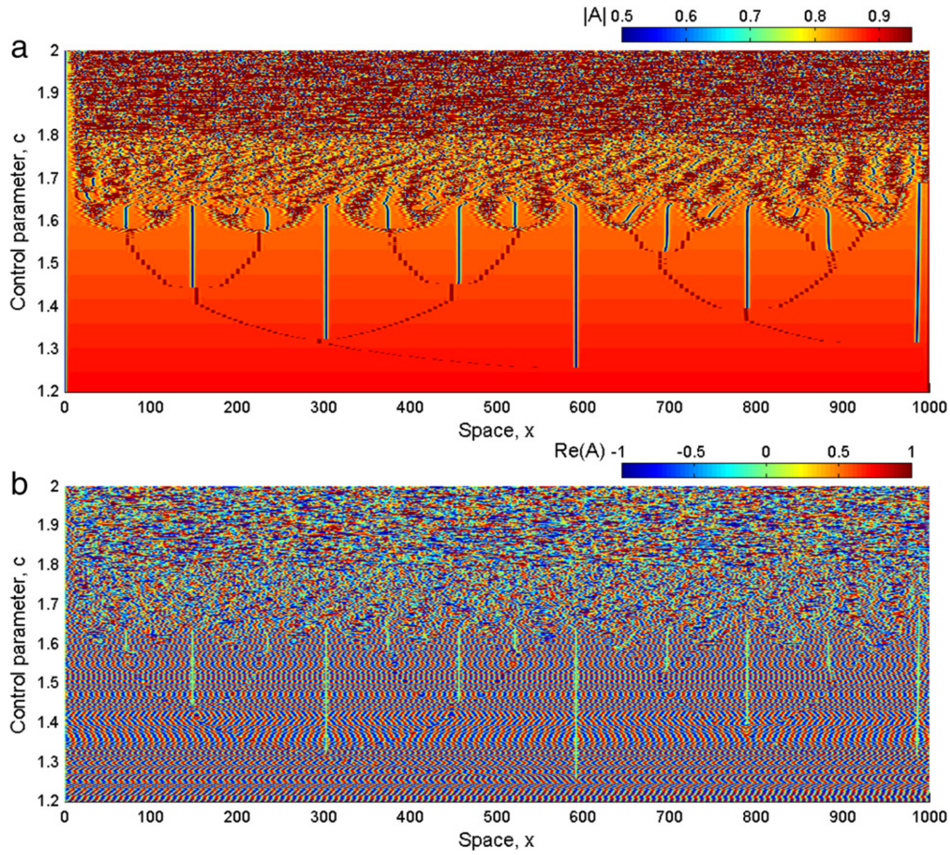

Fig. 1. (a) In this single simulation of (1), a gradual increase in the control parameter c causes a progression from an effectively uniform state in $|A|$ (a plane wave in the real and imaginary parts of A, as is shown for $\operatorname{Re} A$ in (b)) at $c = 1.2$ to full spatiotemporal chaos at $c = 2$. This figure combines the spatial profiles of $|A|$ immediately prior to each increment in c (each increment was 0.001, and increments were made every 3000 time units). Note that values of $|A| \leq 0.5$ are shown in the same colour; this aids visual clarity, since the regions in which $|A| < 0.5$ are very localised in space. See Fig. 2 for alternative view of this data, for specific values of c. (b) Exactly the same simulation as in (a) but instead showing $\operatorname{Re} A$. This simulation took 10 days on an Intel Xenon X5560, 1333 MHz processor, with a 64 bit operating system.

that consists of half of a stationary Nozaki–Bekki hole [24,25], together with a thin boundary layer near $x = L$. When the linear dispersion parameter in the CGLE is zero, the (unique) stationary Nozaki–Bekki hole has a very simple analytical form: $|A| = |A^*| \tanh(x/\sqrt{2})$, $\partial_x \arg A = \sqrt{1 - |A^*|^2} \tanh(x/\sqrt{2})$, where $|A^*(c)| = ([1 + \sqrt{1 + (8/9)c^2}]/2)^{-1/2}$ [22,24]. There is very close agreement between this formula and the numerical solutions of (1) and (2), for $c < 1.110$. Thus the solution is approximately constant in $|A^*(c)|$ and $\partial_x \arg(A^*(c))$, other than very close to the boundaries. In fact, both the real and imaginary parts of $A^*(c)$ exhibit plane waves:

$$A^* = |A^*| \cos(\Phi(x, t)) + i|A^*| \sin(\Phi(x, t)) \qquad (3)$$

where $\Phi(x, t) = K \pm x\sqrt{1 - |A^*|^2} - c|A^*|^2 t$ and K is an arbitrary constant. We refer the reader to [19,22] for details of this mechanism of plane wave generation.

Beyond $c = 1.110$, the plane waves selected by our boundary conditions are no longer stable. In a previous study we showed that when $1.110 < c < 1.576$ the instability is convective [19], meaning that small perturbations to the selected plane wave solution grow in time only while simultaneously moving. In simulations this results in bands of plane waves propagating in alternating directions, separated by localised defects known as "shocks" and "holes". In contrast, when $c > 1.576$, the selected plane waves are absolutely unstable [19], meaning that perturbations to the plane waves grow pointwise. Correspondingly, simulations show irregular spatiotemporal dynamics throughout the domain, rather than plane waves.

This study arose because we observed that when simulating (1) under separated boundary conditions, starting with various initial conditions, the dynamics emerging in simulations when $1.110 < c < 1.576$ were highly sensitive to the initial conditions, the domain length, and the value of c (an example is given in Fig. 1 of [19]). This led us to explore the effects of increasing c only, without resetting the initial conditions. Initially we increased c in relatively large increments and again found the appearance of new defect solutions at unpredictable locations. We then experimented with changing the increment size, and found that the use of sufficiently small increments in c, between time windows of sufficient length to remove transient dynamics, revealed a clear structure to the onset of spatiotemporal chaos (illustrated in Fig. 1). In this study, we therefore set out to understand this emergent structure.

Our work here extends our previous research into the spatiotemporal dynamics observed in simulations of (1) under a variety of initial and boundary conditions [14,19,27,28]. What makes this study different is that we focus on explaining one specific detailed transition to spatiotemporal chaos. In [19] we showed how to calculate the numerical threshold for absolutely unstable plane wave solutions to (1) under the same boundary conditions as studied here. In [27,28] we showed how to calculate the width of the band of plane waves emerging behind propagating fronts, a different plane wave generating mechanism, prior to their subsequent transition to spatiotemporal chaos. There, the initial perturbations to the plane wave bands were introduced by the propagating fronts.

In the scenario studied here, the perturbations to the complete solution (potentially including bands of plane waves) are introduced by the increment in the control parameter c. However, as in