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Minimal atmospheric finite-mode models preserving symmetry and generalized Hamiltonian structures

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ABSTRACT

A typical problem with the conventional Galerkin approach for the construction of finite-mode models is to keep structural properties unaffected in the process of discretization. We present two examples of finite-mode approximations that in some respect preserve the geometric attributes inherited from their continuous models: a three-component model of the barotropic vorticity equation known as Lorenz' maximum simplification equations [E.N. Lorenz, Maximum simplification of the dynamic equations, Tellus 12 (3) (1960) 243–254] and a six-component model of the two-dimensional Rayleigh–Bénard convection problem. It is reviewed that the Lorenz-1960 model respects both the maximal set of admitted point symmetries and an extension of the noncanonical Hamiltonian form (Nambu form). In a similar fashion, it is proved that the famous Lorenz-1963 model violates the structural properties of the Saltzman equations and hence cannot be considered as the maximum simplification of the Rayleigh–Bénard convection problem. Using a six-component truncation, we show that it is again possible to retain both symmetries and the Nambu representation in the course of discretization. The conservative part of this six-component reduction is related to the Lagrange top equations. Dissipation is incorporated using a metric tensor.

1. Introduction

Various models of the atmospheric sciences are based on nonlinear partial differential equations. Besides numerical simulations of such models, it has been tried over the past fifty years to capture at least some of their characteristic features by deriving reduced and much simplified systems of equations. A common way for deriving such reduced models is based on the Galerkin approach: One expands the dynamic variables of a model in a truncated Fourier (or some other) series, substitutes this expansion into the governing equations and studies the dynamics of the corresponding system of ordinary differential equations for the expansion coefficients. Although the number of expansion coefficients is usually minimal to allow for an analytic investigation, these reduced models have been used in order to explain some common properties of atmospheric models.

To the best of our knowledge there is up to now no universal criterion for the selection of modes or the choice of truncation of the series expansion. However, at least some cornerstones for the Galerkin approach are already settled. It is desirable for finite-mode models to retain structural properties of the original set of equations, from which they are derived [1,2]. Such properties are, e.g., quadratic nonlinearities, conservation of energy and one or more vorticity quantities in the nondissipative limit and preservation of the Hamiltonian form.

Recently, an extension of the Hamiltonian structure based on the idea of Nambu [3] to incorporate multiple conserved quantities in a system representation also came into focus. It was shown in [4–9] that various equations of ideal hydrodynamics and magneto-hydrodynamics allow for a formal Nambu representation. It therefore seems reasonable to derive finite-mode models that also retain this structure. Moreover, almost all models in the atmospheric sciences possess symmetry properties. These symmetries should thus be taken into account in low-dimensional modeling too, which is an issue in the field of equivariant dynamical systems (see, e.g., [10]).

The general motivation for this work is that low-order models are still in widespread use in the atmospheric sciences. It has been mentioned above that their original purpose was to identify characteristic features of the atmospheric flow in the pre-supercomputer era. While the advent of supercomputers partially renders this aim obsolete, finite-mode models are still valuable for testing advanced methods in the atmospheric sciences, related to issues of predictability, ensemble prediction, data assimilation or stochastic parameterization [11–14]. Such finite-mode models

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offer the possibility for a conceptual understanding of techniques that are to be used in comprehensive atmospheric numerical models later on. For such testing issues, in turn, it is essential to have finite-mode models that preserve the structure of the underlying set of partial differential equations at least in some minimal way.

In this paper we give two examples of finite-mode models that retain the above mentioned features of their parent model. The first is the three-component Lorenz-1960 model, derived as the maximum simplification of the vorticity equation [15]. The second is a six-component extension of the Lorenz-1963 model [16]. The authors are aware that there exist a great variety of other finite-mode (Lorenz) models, such as e.g. [17–19], possessing richer geometric structure and allowing us to address other important issues in the atmospheric sciences, such as the existence of a slow manifold, atmospheric attractors, balanced dynamics and the initialization problem of numerical weather prediction. Results in these directions can be found, besides in the original papers by Lorenz, e.g., in [20–23]. The choice to investigate the Lorenz-1960 and Lorenz-1963 models, however, is reasonable since the latter still is one of the most prominent finite-mode models used in dynamic meteorology for testing issues as reviewed above. As we are going to show, the Lorenz-1963 model in various respects does not constitute a sound geometric model, so the derivation of a revised version of this system appears to be well justified. The Lorenz-1960 model, on the other hand, has been chosen as it is the simplest system for which the techniques to be applied in this paper can be demonstrated.

The Lorenz-1963 model is a dissipative model and as such it necessarily violates conservative properties. On the other hand this is a rather typical situation for more comprehensive atmospheric numerical models too. Usually, the conservative dynamical core of such models is coupled to a number of dissipative processes such as friction, precipitation and radiation. Nonetheless, it is a necessary condition that the numerics for the dynamical core itself do not violate the structural properties of the underlying conservative dynamics [24]. Any valuable toy model of the atmosphere should reflect this, e.g. by consisting of the superposition of a conservative part and a dissipative part. This is one of the guiding principles for our derivation of the generalized Lorenz-1963 model.

The organization of this paper is as follows: Properties of discrete and continuous Nambu mechanics are briefly reviewed in Section 2. Section 3 includes a description of the Lorenz-1960 model, establishing its Nambu structure and its compatibility with the admitted point symmetries of the barotropic vorticity equation. In Section 4, it is shown that the Lorenz-1963 model is neither compatible with the corresponding Nambu (Hamilton) form of the Saltzman convection equations nor with its point symmetries. We hereafter identify the maximum simplification of the Saltzman convection equations [25] that reflects both symmetries and the proper Nambu structure of the continuous model. Finally, in Section 5 we sum up our results and discuss some open questions.

2. Nambu mechanics

Since Nambu mechanics emerged from discrete Hamiltonian mechanics, it is convenient to start with a short description of the latter. The evolution equation of a general n-dimensional Hamiltonian system is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\},$$

where $F = F(z_i)$ is an arbitrary function of the phase space variables z_i, $i = 1, \dots, n$, H is the Hamiltonian function and $\{.,.\}$ is a Poisson bracket, which satisfies bilinearity, skew-symmetry and the Jacobi identity. For discrete Hamiltonian systems, the Poisson bracket is characterized by an antisymmetric rank two tensor that can depend on the coordinates of the underlying phase space. In modern Hamiltonian dynamics, this tensor is allowed to be singular, leading to the notion of a *Casimir* function C, which Poisson-commutes with all arbitrary functions $G(z_i)$

$$\{C, G\} = 0, \quad \forall G.$$

Setting $G = H$, it follows that every Casimir is in particular also a conserved quantity.

Guided by Liouville's theorem stating volume-preservation in phase space, Nambu [3] proposed a formalism for discrete mechanical systems allowing *multiple* conserved quantities to determine, at the same level of significance, the evolution of a dynamical system. More precisely, let us consider a point mechanical system with n degrees of freedom and $n-1$ functionally independent conserved quantities H_j, $j = 1, \dots, n-1$. The evolution equation for an arbitrary function F according to Nambu is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial(F, H_1, H_2, \dots, H_{n-1})}{\partial(z_1, z_2, \dots, z_n)} =: \{F, H_1, H_2, \dots, H_{n-1}\}.$$

The above bracket operation is called a *Nambu bracket*, which due to the properties of the Jacobian is non-singular, multi-linear and antisymmetric under the exchange of any two arguments. It was demonstrated in [26] that a Nambu bracket also fulfills a generalization of the Jacobi identity, which reads

$$\begin{aligned}&\{\{F_1, \dots, F_{n-1}, F_n\}, F_{n+1}, \dots, F_{2n-1}\}\\ &\quad + \{F_n, \{F_1, \dots, F_{n-1}, F_{n+1}\}, F_{n+2}, \dots, F_{2n-1}\}\\ &\quad + \cdots + \{F_n, \dots, F_{2n-2}, \{F_1, \dots, F_{n-1}, F_{2n-1}\}\}\\ &\quad = \{F_1, \dots, F_{n-1}, \{F_n, \dots, F_{2n-1}\}\}\end{aligned} \tag{1}$$

for any set of $2n-1$ functions F_i. Various discrete models that allow for a Nambu formulation were identified, e.g. the free rigid body [3], a system of three point vortices [5], and the conservative Lorenz-1963 model [27], which is discussed in some detail below.

It appears that the application of ideas of discrete Nambu mechanics to field equations was first considered in [28] (and even earlier in a talk [29]), and later independently by Névir and Blender [6]. It was noted that the singularity of many continuous Poisson brackets of fluid mechanics may be formally removed by extending them to tribrackets using explicitly *one* of their Casimir functionals as an additional conserved quantity. That is, despite the fact that partial differential equations represent systems with infinitely many degrees of freedom, up to now there only exist models using one additional conserved quantity. This way, the term *continuous Nambu mechanics* (referring to a Nambu representation of field equations) is at once misleading, though it is already used in several papers.

The restriction to tribrackets may be traced back to the underlying Lie algebras on which the Poisson brackets in Eulerian variables are based on [30,31]. Hence, the fixed relation between the dimension of the phase space and the number of conserved quantities used for a system representation is lost in continuous Nambu mechanics. In the atmospheric sciences, this generalization is called *energy-vorticity theory*, as the employed Casimir functional is frequently related to some vortex integral. Since in the atmospheric sciences the evolution of the rotational wind field is dominant over different scales, the energy-vorticity description may be well suited for a better understanding of e.g. turbulence. Among others, models that can be cast into energy-vorticity form include the inviscid non-divergent 2d and 3d barotropic vorticity equations, the quasi-geostrophic potential vorticity equation and the governing equations of ideal fluid mechanics as well as equations of magnetohydrodynamics [4,5,32,7,8].

The main problem with continuous Nambu mechanics is that it is up to now not clear whether it is possible to state an appropriate