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Journal of Geometry and Physics 55 (2005) 316–352

JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Geometrical origin of the $*$-product in the Fedosov formalism

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Received 19 May 2004; received in revised form 16 December 2004; accepted 18 December 2004
Available online 5 February 2005

Abstract

The construction of the $*$-product proposed by Fedosov is implemented in terms of the theory of fibre bundles. The geometrical origin of the Weyl algebra and the Weyl bundle is shown. Several properties of the product in the Weyl algebra are proved. Symplectic and abelian connections in the Weyl algebra bundle are introduced. Relations between them and the symplectic connection on a phase space M are established. Elements of differential symplectic geometry are included. Examples of the Fedosov formalism in quantum mechanics are given.

MSC: 53C07; 53D55

PACS: 02.40.Hw; 03.65.Ca

JGP SC: Deformation quantization

Keywords: Weyl bundle; Connection in a vector bundle; Deformation quantization; Fedosov formalism

1. Introduction

The standard formulation of quantum mechanics in terms of a complex Hilbert space and linear operators is mainly applied for systems, whose classical limit may be de-

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0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2004.12.010

scribed on a phase space $\mathbb{R}^{2n}$. There are two reasons for this situation: the fundamental operators $\hat{q}^i, \hat{p}_i, i = 1, \ldots, 2n$, are well defined by the Dirac quantization scheme [1] only in the case of $\mathbb{R}^{2n}$, and the operator orderings are based on the Fourier transform [2–4].

Unless we agree to weaken the foregoing assumptions [5] to deal with other quantum systems it is necessary to use geometric quantization [6,7] or deformation quantization.

Classical mechanics is a physical theory which works perfectly on arbitrary differential manifolds. From this reason shortly after presenting a standard version of quantum theory, researchers began to look for an equivalent formulation of quantum mechanics based on differential geometry. The first complete version of quantum theory in the language of the theory of manifolds appeared in the middle of the XXth century, when Moyal [8] using previous works by Weyl [9], Wigner [10] and Groenewold [11] presented quantum mechanics as a statistical theory. His results are only valid for the case $\mathbb{R}^{2n}$. However, the paper by Moyal contains the seminal ideas about deformation quantization, since the main result of this work is the substitution of the point-wise product of functions in phase space for a new product which is a formal power series in $\hbar$.

A modern version of Moyal's deformation quantization on an arbitrary differential manifold was proposed by Bayen et al. [12] in 1978. The mathematical structure of this formulation of the quantum theory is based, like Hamiltonian classical mechanics, on differential geometry of symplectic spaces. Observables are smooth real functions on a phase space and states are represented by functionals. Macroworld appears in this formalism as the limit of the quantum reality for the Planck constant $\hbar$ tending to 0^+.

The list of axioms constituting deformation quantization looks as follows:

(i) a state of a physical system is described on a $2n$-dimensional phase space M,
(ii) an observable is a real smooth function on M,
(iii) for every complex-valued smooth functions (f, g, h) of $C^\infty(M)$ the $*$-product fulfills the following conditions:

(a) $$f * g = \sum_{t=0}^{\infty} \left(\frac{i\hbar}{2}\right)^t M_t(f, g),$$

where $M_t(\cdot, \cdot)$ is a bidifferential operator on M (see definition later, formula (1.1)).

(b) The element M_0 represents the 'usual' commutative product of functions i.e.

$$M_0(f, g) = f \cdot g.$$

Thus, at the classical limit

$$\lim_{\hbar \to 0^+} f * g = f \cdot g.$$

(c) The quasi-Dirac quantization postulate holds

$$M_1(f, g) - M_1(g, f) = 2\{f, g\}_P,$$

where $\{\cdot, \cdot\}_P$ stands up for Poisson brackets.

(d) Associativity also holds

$$\sum_{t+u=s} (M_t(M_u(f, g), h) - M_t(f, M_u(g, h))) = 0 \quad \forall s \geq 0.$$