DYNAMICS OF AFFINELY-RIGID BODIES WITH DEGENERATE DIMENSION

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Dynamical models with degrees of freedom ruled by linear and affine groups have been discussed in various aspects by many people, e.g. O. I. Bogoyavlensky [7] and J. J. Sławianowski [4, 5]. We concentrate on models with degenerate dimension, when the configuration space consists of injections from $\mathbb{R}^m$ into $\mathbb{R}^n$. Equations of motion are derived with special stress on the stationary solutions (stationary ellipses). The isotropic models are related in an interesting way to the theory of Grassmann and Stiefel manifolds.

Keywords: affinely-rigid bodies, linear groups, homogeneous spaces, stationary solutions, Grassmann and Stiefel manifolds.

1. Introduction

Mechanics of affinely-rigid body was developed in a series of publications [1, 2, 4–8] within the framework both of Hamiltonian and dissipative models. Roughly speaking, affinely-rigid body is a system of material points (discrete or continuous) moving in such a way that all affine relations between its constituents remain invariant, thus, the material straight lines continue to be straight lines, their parallelism is conserved, and the ratios of segments on the same material straight line are conserved. In the mentioned publications the problem was discussed, generally, in n dimensions, although, of course, only the special cases $n = 1, 2, 3$ are of direct physical interest. But from the purely mathematical point of view it is just more convenient to study the general situation, i.e. for an arbitrary value of n. Let us mention, incidentally that there exists an interesting and unexpected link between some n-dimensional models of affinely-rigid body and one-dimensional n-body problems, in particular, with the dynamics of integrable lattices studied by Calogero [9], Moser [10, 11], Sutherland, and others.

The model of systems with affine constraints is applicable in a wide spectrum of problems, like macroscopic elasticity (when the length of excited waves is of the same order as the body size; a kind of tidal motion), media with microstructure (e.g. the Eringen micromorphic continua), nuclear and molecular dynamics, and dynamics of astrophysical objects, including the theory of the shape of the Earth. The last problem has a very long history starting from Newton himself through Riemann, Kronecker and Dedekind till Chandrasekhar [6] almost recently. Obviously, the microscopic problems require quantum treatments.

Various mathematical aspects of the problem were deeply investigated by M. Roberts, C. Wulff [12, 13], A. A. Burov [16, 17], S. Ya. Stepanov [17], D. P. Chevallier [14, 16], and others. This covers, in particular, problems like relative equilibria and relative periodic orbits.

Very interesting aspects of affine motion within the context of hydrodynamics and astrophysics were studied by O. I. Bogoyavlenskij [7]. There were used some very sophisticated methods of the theory of dynamical systems. Astrophysical studies made by O. I. Bogoyavlenskij had to do with slightly different problems than our affine constraints imposed on the system motion. Namely, he was searching for affine-modes solutions of unconstrained problems.

2. General formulation

Affine motion consists of spatial translations, rotations, and homogeneous deformations. On the purely kinematical level the metric tensor is superfluous when defining degrees of freedom. The physical space may be metrically amorphous and endowed only with affine (Tales) geometry. The metric tensor becomes essential when we construct dynamical models, although, as we show, there exist also metric-free models. They present at least academic mathematical interest and might seem relatively exotic from the physical point of view. Nevertheless some physical applications are not excluded [8].

When some Cartesian coordinates and reference coordinates are fixed, the configuration space of an n-dimensional affinely-rigid body may be identified with the affine group

$$Q = \mathrm{GAf}\,(n, \mathbb{R}) \simeq \mathrm{GL}\,(n, \mathbb{R}) \otimes_s \mathbb{R}^n .$$

In the above semi-direct product the factor $\mathbb{R}^n$ refers to the center of mass motion, i.e. translational motion (physical space identified with $\mathbb{R}^n$), whereas

$$Q_{\mathrm{int}} = \mathrm{GL}\,(n, \mathbb{R}) ,$$

i.e. the general linear group describes internal degrees of freedom (one says rather about degrees of freedom of the relative motion in mechanics of extended systems). In the case of continuous bodies one should use rather the connected component of unity in $\mathrm{GL}(n, \mathbb{R})$, i.e. the group of proper linear transformations $\mathrm{GL}^+(n, \mathbb{R})$ (positive determinants).

In more sophisticated terms, it is customary to use two logically different affine spaces (M, V) and (N, U), i.e. respectively, the physical space M and the material space N. Here V and U denote, respectively, the linear space of translations in M and N. The configuration space is identified with the manifold $Q = \mathrm{AfI}\,(N, M)$ of affine isomorphisms of N onto M. The co-moving mass distribution is described by a fixed, time-independent positive measure μ on N. When $\mu(N) < \infty$, the