DIRAC–NIJENHUIS STRUCTURES ON LIE BIALGEBROIDS*

LIU BAO-KANG and HE LONG-GUANG

Department of Mathematics, Capital Normal University, Beijing 100037, China
(e-mails: liubk@mail.cnu.edu.cn, helg@mail.cnu.edu.cn)

(*Received July 24, 2004 — Revised January 3, 2005*)

This paper is a continuation of the work done in [1]. We define Dirac–Nijenhuis structures (DN structures) on Lie bialgebroids, a generalization of the notion of Dirac–Nijenhuis structures on manifolds given in [1]. However, the problems we are going to deal with are more complex than the ones in [1]. The necessary and sufficient conditions for a structure to be a DN structure on a Lie bialgebroid as well as several examples of these structures are given. At the end of the paper we discuss the fundamental sections of DN structures.

1991 Mathematics Subject Classifications: 58F05, 17B66, 22A22, 53C99, 58H05.

Keywords: Dirac structure, Nijenhuis tensor, deformation, Lie bialgebroid, Dirac–Nijenhuis structure, fundamental section.

1. Introduction

In their work [2], F. Magri and C. Morosi made deep studies of Poisson–Nijenhuis structures. Later Y. Kosmann-Schwarzbach and F. Magri in [3] simplified the calculations given in [2]. T. J. Courant introduced the notion of Dirac manifold in [4] and discussed its properties. We have given in [1] a compatibility condition for Dirac structures and Nijenhuis tensors on manifolds and have defined the concept of Dirac–Nijenhuis structures (DN structures) on manifolds. In this way we generalized the notion of PN manifold and of ΩN manifold. In our present paper we are going to generalize the notion of DN structures on manifolds to the case of Lie bialgebroids and discuss some of their properties. Hence this paper is closely related to [1]. Since the canonical Lie algebroid TP, the tangent bundle of a manifold P, and the canonical Lie bialgebroid (TP, T^*P) are the special cases of general Lie algebroid A and Lie bialgebroid (A, A^*), respectively, all the results that follow will be more complex than the ones we have obtained for DN manifolds. The former include the latter as its special case.

Recently, J. Clemente-Gallardo and J. M. Nunes da Costa [7] discussed DN structures from another point of view. The notion of DN structures, the most important concept defined in [7], is quite different from ours in [1]. They first

*Lecture given at the 36th Symposium on Mathematical Physics, Toruń, June 9–12, 2004.

define trivial deformation of a Lie bialgebroid $((A, [\ ,\ ], a), (A^*, [\ ,\ ]_*, a_*))$ by a pair $\mathcal{N} = (N, \Upsilon)$, where N is a Nijenhuis tensor on A and Υ a Nijenhuis tensor on A^*. The triviality of the deformation means that $((A, [\ ,\ ]_N, a \circ N), (A^*, [\ ,\ ]_{*\Upsilon}, a_* \circ \Upsilon))$ is a new Lie bialgebroid. Then the authors define trivial deformation of the double of a Lie bialgebroid. By a trivial deformation of the double $(E = A \oplus A^*, [\ ,\ ], (\ ,\ )_+, \rho = a + a_*)$ they mean that the deformed double $(E = A \oplus A^*, [\ ,\ ]_{\mathcal{N}}, (\ ,\ )_+, \rho \circ \mathcal{N})$ coincides with the double $(E = A \oplus A^*, [\ ,\ ]^{\mathcal{N}}, (\ ,\ )_+, \rho^{\mathcal{N}} = a \circ N + a_* \circ \Upsilon)$ of the deformed Lie bialgebroid $((A, [\ ,\ ]_N, a \circ N), (A^*, [\ ,\ ]_{*\Upsilon}, a_* \circ \Upsilon))$. It is easy to check that $\rho \circ \mathcal{N} = \rho^{\mathcal{N}}$. It follows that the coincidence is equivalent to $[\ ,\ ]_{\mathcal{N}}=[\ ,\ ]^{\mathcal{N}}$. Then they consider two types of deformations of Dirac structures. In the deformation of type 1 they emphasize trivial deformations of the double of a Lie bialgebroid and consider the conditions for Dirac structures on the original Lie bialgebroid to be Dirac structures on the deformed Lie bialgebroid. In the deformation of type 2 they do the same for the Dirac structures but drop the triviality of the deformations of the double of the Lie bialgebroid. In both cases deformation of a Dirac structure D means the deformation of the Lie bialgebroid on which D is defined while D itself remains the same throughout [7]. In other words, to deform D means to deform the structure of the Lie bialgebroid. Another requirement relating the deformation of Dirac structures in both cases is that $\mathcal{N}$ is a Nijenhuis operator on the Lie algebra $(\Gamma(D), [\ ,\ ])$, a condition satisfied by the first case but not by the second one unless some additional conditions are satisfied.

In the last section of [7] the authors define two types of DN structures $(D, \mathcal{N})$ on a Lie bialgebroid (A, A^*) and the second type obviously generalizes the first one. Let D be a Dirac structure on (A, A^*) and $\mathcal{N} = N \times \Upsilon : A \oplus A^* \longrightarrow A \oplus A^*$ be an operator preserving D (i.e. $\mathcal{N}(D) \subset D$), where N and Υ are Nijenhuis operators on A and A^*, respectively. Then $(D, \mathcal{N})$ is defined to be a DN structure of type 1 if:

(1) $\mathcal{N}$ is a trivial deformation of the double $E = A \oplus A^*$ of (A, A^*) and the Nijenhuis torsion of $\mathcal{N}$ with respect to the bracket defined on $\Gamma(E)$ vanishes.

(2) D is a Dirac structure with respect to the deformed bracket $[\ ,\ ]_{\mathcal{N}}$.

$(D, \mathcal{N})$ is defined to be a DN structure of type 2 if conditions (1) and (2) are replaced by

(1′) $\mathcal{N}$ is a Nijenhuis operator on the Lie algebra $(\Gamma(D), [\ ,\ ])$.

In our present paper we shall deal with some problems similar to those in [7], but we shall do that in a different way. As the first step we introduce a pair of compatible Nijenhuis tensors N_1 and N_2 on A. Then, under certain condition, we deform (A, A^*) by the Nijenhuis tensor $N = N_1N_2$ to obtain a deformed Lie bialgebroid $(A', A^*) = ((A, [\ ,\ ]' = [\ ,\ ]_N, a' = a \circ N), (A^*, [\ ,\ ]_*, a_*))$. Note that only the first Lie algebroid is deformed while the second one is not. From here we see the first remarkable difference with [7] in which both Lie algebroids are deformed. Another difference relates to the way of deforming Dirac structures. A natural way of deforming a Dirac structure L on (A, A^*) is deforming vectors in L by N_1 and covectors by N_2, i.e. we define a deformed Dirac structure L_1