Wave Motion 51 (2014) 581–588

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Long-wave asymptotic theories: The connection between functionally graded waveguides and periodic media

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HIGHLIGHTS

- The connection between the long-wave asymptotic theories for waveguides and periodic media is discussed.
- Similarity of trapped modes for both cases is emphasised.
- Both high-frequency and low-frequency regimes are addressed.

ARTICLE INFO

Article history:
Received 5 February 2013
Received in revised form 7 September 2013
Accepted 26 September 2013
Available online 16 October 2013

Keywords:
Asymptotic
Low-frequency
High-frequency
Homogenisation
Waveguide
Functionally graded

ABSTRACT

This article explores the deep connections that exist between the mathematical representations of dynamic phenomena in functionally graded waveguides and those in periodic media. These connections are at their most obvious for low-frequency and long-wave asymptotics where well established theories hold. However, there is also a complementary limit of high-frequency long-wave asymptotics corresponding to various features that arise near cut-off frequencies in waveguides, including trapped modes. Simultaneously, periodic media exhibit standing wave frequencies, and the long-wave asymptotics near these frequencies characterise localised defect modes along with other high-frequency phenomena. The physics associated with waveguides and periodic media are, at first sight, apparently quite different, however the final equations that distill the essential physics are virtually identical. The connection is illustrated by the comparative study of a periodic string and a functionally graded acoustic waveguide.

1. Introduction

For long waves within a waveguide, at low frequency, intuition suggests that (for waveguides governed by the Helmholtz equation with Neumann boundary conditions) the waveguide behaves effectively as a string; when viewed from afar the guide is long and thin. Similarly, a string composed of periodic elements where the length scales associated with the periodicity are much less than the wavelength of the excitation also intuitively behaves as some effective string. In both cases the word "effective" is rather vague, but as we shall see this can be made precise through an asymptotic approach involving two scales; the thickness of the guide or periodicity scale and the length-scale of the guide or overall string. As we shall see an application of multiple scales leads rapidly to an effective equation for these two problems that can be simultaneously treated.

Perhaps less obviously one can also consider high-frequency wave propagation which almost immediately equates to short wavelength, as one typically thinks of waves within a bulk medium, and asymptotic techniques for waveguides based

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http://dx.doi.org/10.1016/j.wavemoti.2013.09.007

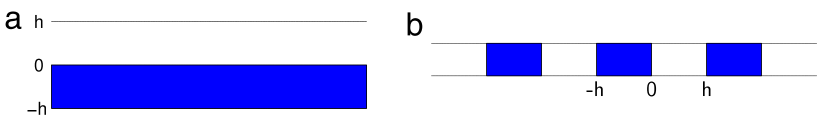

Fig. 1. (a) Functionally graded waveguide and (b) periodic string.

upon the WKBJ ansatz are popular and versatile [1–4]. However, the imposition of boundaries such as those of a waveguide can alter this intuitive viewpoint and long-wave solutions are also possible. Taking a straight, constant thickness, isotropic waveguide a natural approach is to seek modal solutions and create dispersion curves. As is well-known (e.g. see [5]) there are an infinite set of discrete modes each with a cut-off frequency. For each mode the cut-off frequency delineates evanescent, exponentially decaying, solutions from the propagating modes. If one is exactly at the cut-off frequency (possibly at high-frequency) then this is called thickness resonance and the wave simply bounces back and forth across the waveguide and neither propagates to the left or right. The wavelength of this mode along the guide is actually infinite and close to cut-off the wavelength is therefore large. This observation motivates a general asymptotic methodology, summarised in the books [6–8] in the context of thin elastic structures of arbitrary shape. In particular for the case of flat or axisymmetric waveguides with either weakly bent, bulging or thinning walls, [9–11] ordinary differential equations (ODEs) for trapped modes (solutions with finite energy that decay exponentially at infinity) emerge. A key point is that in the deformed region one can shift the local cut-off frequency such that waves propagate locally, but are cut-off away from this region, thereby trapping modes. The simple ODE representation is then very powerful compared to large scale numerical eigenvalue calculations that lack insight.

A complementary, and apparently disconnected, area in wave propagation is that of waves passing through periodic media; this is important in solid-state physics [12], photonics [13] and the emergent areas of metamaterials [14]. For infinite perfectly periodic media, consisting of elementary cells that repeat, one can focus attention on a single elementary cell; quasi-periodic Floquet–Bloch boundary conditions describe the phase-shift as a wave moves through the material and dispersion relations are then deduced that relate the Bloch wavenumber, the phase-shift, to frequency. The eigensolutions that emerge are the Bloch modes, and when these eigensolutions are perfectly in-phase or out-of-phase across the cell then standing waves exist and the frequencies are then called standing wave frequencies (these frequencies can be high). There exist bands of frequencies, called band-gaps, in which propagating Bloch modes do not exist and in which the modes are evanescent. If the perfectly periodic lattice is perturbed then localised defect states can occur, these exponentially decay with distance, and the behaviour is eerily reminiscent of the trapped modes in a waveguide. Indeed at these high standing wave frequencies one can have "thickness resonance" within each elementary cell and the real wavenumber (not the Bloch wavenumber) is infinite as the wave itself is not propagating left or right. Asymptotic techniques based around high-frequency long-wave asymptotics have recently been developed [15] and ODEs in 1D periodic media (or PDEs in 2D) again emerge; this approach also works for microstructured discrete [16] or frame-like media [17]. The basic idea for periodic media is to replace the complicated microstructured medium with an equivalent, effective, continuum on a macro-scale, that is, one wishes to homogenise the medium even when the wavelength and microstructure may be of similar scales.

We will illustrate the connection between the waveguide problem and periodic media by considering, in parallel, two model problems: A functionally graded acoustic waveguide and a periodic piecewise string. These are algebraically completely tractable and are explicitly solved in Section 2. The asymptotics of the low frequency model follow in Section 3 and the linear asymptote emerges together with an effective string equation for both examples. The less intuitive case of high frequencies is dealt with in Section 4, there are only minor differences between the two examples but both feature a rapidly oscillating solution on one scale modulated by a long-scale function that satisfies an effective equation posed entirely upon the long scale. Explicit asymptotic results are found and compared with exact dispersion relations. The asymptotic techniques are also applied to deformed layered waveguides (Section 5). Finally, we gather together some concluding remarks in Section 6.

2. Formulation

2.1. Functionally graded waveguide

Let us consider a straight waveguide, of constant width, in $|y| < h$ and $|x| < \infty$ with Neumann boundary conditions upon the waveguide walls, see Fig. 1(a). If the waveguide is elastic and excited by out-of-plane oscillations, so there is shear-horizontal (SH) polarisation, then the governing equation is that of acoustics with

$$\left(\nabla^2 + \frac{\omega^2}{\hat{c}^2(y/h)}\right) u(x, y) = 0. \tag{1}$$

Notably we have allowed the wave speed, $\hat{c}$, to vary across the waveguide and we take a reference wave speed to be $c_0 = \hat{c}(0)$ so $\hat{c}(y/h) = c_0 c(y/h)$; the boundary condition is that $\partial u/\partial y = 0$ on $y = \pm h$. In (1) the frequency is denoted by ω.