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Wave trapping in a two-dimensional sound-soft or sound-hard acoustic waveguide of slowly-varying width

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In this paper we derive novel approximations to trapped waves in a two-dimensional acoustic waveguide whose walls vary slowly along the guide, and at which either Dirichlet (sound-soft) or Neumann (sound-hard) conditions are imposed. The guide contains a single smoothly bulging region of arbitrary amplitude, but is otherwise straight, and the modes are trapped within this localised increase in width.

Using a similar approach to that in Rienstra (2003) [13], a WKBJ-type expansion yields an approximate expression for the modes which can be present, which display either propagating or evanescent behaviour; matched asymptotic expansions are then used to derive connection formulae which bridge the gap across the cut-off between propagating and evanescent solutions in a tapering waveguide. A uniform expansion is then determined, and it is shown that appropriate zeros of this expansion correspond to trapped mode wavenumbers; the trapped modes themselves are then approximated by the uniform expansion. Numerical results determined via a standard iterative method are then compared to results of the full linear problem calculated using a spectral method, and the two are shown to be in excellent agreement, even when ϵ, the parameter characterising the slow variations of the guide's walls, is relatively large.

1. Introduction

It is well-established that vibrational energy can become trapped within waveguides by local changes in the guide's width (e.g., [1,2]) or curvature (e.g. [2–5]), resulting in what are termed trapped modes: localised solutions of the homogeneous time-harmonic boundary-value problem. In this paper, we focus on the trapping that can occur within a straight two-dimensional acoustic waveguide with a localised increase in width (i.e., a *bulge*). Physically, this geometry is capable of trapping waves since a particular mode may be evanescent in the narrower uniform region to either side of the bulge, but propagating in a uniform guide of width equal to the bulge width. Thus it is possible for a solution to exist which is oscillatory within the bulge, but decaying outside it, i.e., a trapped wave, the trapped mode frequency then lying between the cut-off frequencies associated with the width of the bulge and the width of the surrounding straight region.

Analytical determination of trapped wave frequencies and the associated modal structure is generally difficult, but if the width of the waveguide is slowly-varying, in the sense that the length-scale ϵ^{-1} over which the width changes satisfies $0 < \epsilon \ll 1$, then this small parameter can be used to develop an asymptotic scheme. In a recent series of papers [1,2], an asymptotic procedure is developed which allows calculation of the trapped wave frequencies (or, more precisely, the $O(\epsilon)$ correction away from the cut-off frequencies) as solutions of a simple ODE eigenvalue problem, given the additional

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geometrical constraint that the amplitude of the bulge is $O(\epsilon^2)$. (This amplitude scaling means that the solution's change in character from propagating to evanescent is not localised.) The ODE eigenvalue problem is then accurately and efficiently solved using a spectral method.

A complementary problem to determining trapped modes within a slowly-varying waveguide is to instead determine an approximation to the types of *propagating* modes which can exist therein. This is commonly-achieved using variations on the general WKBJ ansatz $\phi = Ae^{iP/\epsilon}$ (see [6–8] for examples of using the approximation in problems in which the curvature rather than the width varies slowly). The approximation was used in [9] to model surface gravity water waves above a slowly-varying bed, and modified in [10] to include an expansion of the phase P in powers of ϵ. In [6–8], A and P are expanded in powers of ϵ and are functions of both longitudinal and transverse coordinates, which allow A and P to be identified as the real amplitude and phase, respectively. The expansion of P also has the effect of allowing the approximation to be uniform in range. The expressions derived are referred to as *quasi-modes* in [6–8], are uncoupled, and to first order coincide with the adiabatic approximation in which gradients in waveguide width are ignored, and the modes are given locally by the separable solutions which exist in a uniform guide of the same local width.

A related body of literature exists concerning sound propagation in three-dimensional slowly-varying lined flow ducts (e.g. [11–14]). In [11–13], a multiple scales analysis reveals the form of the modes which can be present in the duct, and in [13] connection formulae are derived for the turning point at the transition between propagating and evanescent behaviour. In [14], a uniform expansion is derived for a mode undergoing cut-on cut-off transition.

In this paper, we use an expansion similar to that which yields the quasi-mode expressions to instead examine the trapping problem. We first derive a version of quasi-modes which allows both propagating and evanescent behaviour, depending on whether the wavenumber is greater or less than the local cut-off. The resulting expression is similar to that derived in [11–13] for sound propagation in three-dimensional slowly-varying lined flow ducts. These expressions stem from a WKBJ-type ansatz $\phi = Ae^{P}$, with A and P both functions of longitudinal and lateral coordinates, and A expanded as $A = A_0 + \epsilon A_1 + \cdots$ but P written as simply $P = \epsilon^{-1}P_{-1}$, since higher-order terms in the expansion of P can be absorbed into A. The "phase" P is allowed to be real or imaginary to produce either propagating or evanescent behaviour.

Now, a mode trapped within a bulge is oscillatory in nature within the centre of the bulge, and then changes in character to an evanescent wave as the narrower portion of the guide is reached, the point at which this character change occurs being an example of a turning point (see, for example [15]), at which in particular the expressions derived for the quasi-modes are not valid. However, formulae which connect the propagating and evanescent waves across the turning point can be obtained via the method of matched asymptotic expansions (in a fashion similar to that used in [13]). Motivated by the solution appropriate in the vicinity of the turning point, a uniformly valid expansion is then derived which includes the two quasi-mode forms as limiting behaviour. The ansatz adopted to derive this uniform expansion is similar to that used to calculate higher order uniform approximations to ODE turning point problems (see [15]), and the resulting expansion is also similar to that derived in [14]. Determining appropriate zeros of this uniform expansion via a standard iterative procedure then furnishes highly accurate approximations to the trapped wave frequencies and modal structures.

We note that the strengths of the procedure presented here include that the amplitude of the bulge is arbitrary (so long as the guide's walls remain slowly-varying); the determination of the trapped wavenumbers is ultimately straightforward, simply requiring the application of a standard iterative scheme to a single nonlinear equation, and once the trapped wavenumbers are determined, the uniform expansion immediately offers an analytic expression for the structure of the eigenmode.

The paper proceeds as follows. Throughout, we derive results for the sound-soft problem and then state the corresponding sound-hard result. In Section 2, we derive the quasi-mode expressions which allow both propagating and evanescent behaviour. Then we consider the reflection of one such propagating quasi-mode at a taper in a waveguide, first using a matched asymptotics procedure to connect the propagating and evanescent expansions, and then via a uniform expansion.

In Section 3, we show how the uniform approximation to the taper problem can be used to derive approximations to the corresponding eigenvalue problem, and then compare a selection of these results to numerical approximations to the solution of the full linear problem for the sound-soft case, calculated using a spectral method. Finally, in Section 4 we conclude and offer some suggestions for further work.

2. Waves in a two-dimensional acoustic waveguide of slowly-varying width

2.1. Preliminaries

We wish to determine the values of $\bar{k}$ giving a non-trivial solution to the homogeneous boundary-value problem

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} + \bar{k}^2\bar{\phi} = 0 \quad (-\infty < \bar{x} < \infty, -\bar{h}_-(\bar{x}) < \bar{y} < \bar{h}_+(\bar{x})), \tag{2.1}$$

$$\bar{\phi} \to 0 \quad \text{as } \bar{x} \to \pm\infty, \tag{2.2}$$

supplemented by either the Dirichlet (sound-soft) boundary conditions

$$\bar{\phi} = 0 \quad (-\infty < \bar{x} < \infty, y = \pm\bar{h}_\pm(\bar{x})) \tag{2.3}$$