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A discussion of the properties of the Rayleigh perturbative solution in diffraction theory

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Abstract

The problem of two-dimensional plane-wave scattering from an infinite periodic profile with a Dirichlet boundary condition is considered. It is shown that the Rayleigh theory yields the same perturbation series for the reflection coefficients as the extinction-theorem formalism. This identity holds up to any order if the boundary profile is infinitely continuously differentiable, and at least up to some finite order if the boundary profile is finitely continuously differentiable. The nature of the derivation proposed allows the extension of these results to all linear homogeneous media encountered in electromagnetism and elastodynamics, to all usual linear boundary conditions, and to the case of scattering from finite bodies. It is indicated that, for interfaces which are periodic finite linear combinations of sinusoids, the Rayleigh perturbative solution is identical to the perturbative solutions obtained with any other approach, and verifies the reciprocity relationships.

Keywords: Diffraction; Perturbation methods

1. Introduction

Among the various approaches used to deal with reflection and transmission of waves at rough interfaces [1,2], the Rayleigh methods are simple to implement. They are based on the hypothesis, postulated by Rayleigh [3], that the field scattered from an interface can be represented by the same sum of outgoing and evanescent waves in all of an half-space bounded by the interface; the amplitudes of those waves are numerically determined by projecting the boundary conditions on a given complete set of functions. The Rayleigh methods turn out to give good numerical results well beyond the limited domain of validity of the Rayleigh hypothesis [4,5]. An explanation of this fact has been indicated for an interface separating isotropic solid media [6] and later given in detail for all homogeneous linear media encountered in electromagnetism and elastodynamics [7]. Other references on the rich and long-standing debate on the Rayleigh hypothesis can be found in Refs. [1,2,7] for instance.

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Contrary to the Rayleigh method, the extinction-theorem method has no theoretical limitation and is therefore regarded as exact. In this approach, the null-field equations [1,2] are solved for the unknown surface fields and the result is used to compute the reflected and transmitted fields. It is formally possible to evaluate the reflection and transmission coefficients by the application of a perturbative approach to either method; the unknown surface fields and the scattered amplitudes are then sought in the form of an integral-power series in the surface elevation [8].

The issue of the identity of these formal series has been investigated in the literature. Jackson et al. [8] have shown for the Dirichlet problem that the five initial terms of the Rayleigh perturbation series are identical to those found with the extinction-theorem perturbative approach. For the same problem, Roginsky [9] has shown that the terms of the perturbation series of the two theories are related through the reciprocity transformation to all orders. In the case of electromagnetic scattering at an analytic interface separating linear homogeneous isotropic media, Bruno and Reitich [10,11] have shown that there exists a domain of sufficiently small surface elevation where the Rayleigh hypothesis holds; they have also shown that the scattered field is analytic in the surface elevation up to and beyond the surface. These results are of fundamental importance as they justify both the Rayleigh perturbative approach and perturbative approaches based on the preliminary evaluation of unknown surface fields; the integral-power series in the surface elevation of the reflection and transmission coefficients derived from these methods must therefore coincide and be invariant under reciprocity. The problem of scattering from finite bodies has recently been investigated by Skaropoulos and Chrissoulidis [12] in the case of plane-wave scattering in an isotropic medium from a soft circular surface with sinusoidal roughness; the scattered wave amplitudes given by the Rayleigh and extinction-theorem perturbative methods are numerically compared and found to be identical at any order considered.

In the present work, we propose a demonstration of the identity of the formal perturbative solutions of the Rayleigh and extinction-theorem formalisms in the case of two-dimensional plane-wave scattering from a periodic boundary with a Dirichlet condition. This identity holds up to any order if the boundary profile is infinitely continuously differentiable, and at least up to some finite order if the boundary profile is finitely continuously differentiable. In the discussion of these results, it will appear that the derivation proposed can be easily extended to all linear homogeneous media encountered in electromagnetism and elastodynamics, to all usual linear boundary conditions, and to finite bodies.

The paper is organized as follows. The two-dimensional quasi-periodic Dirichlet problem is presented in Section 2. The Rayleigh perturbative method is presented in Section 3. The Rayleigh perturbation series for the reflection coefficients are shown to be identical to the extinction-theorem perturbation series in Section 4. We discuss and generalize the above results in Section 5.

2. Problem description and fundamental relations

We consider a one-dimensional periodic profile (S) defined by the relation $z = hf(x)$, where x and z are the cartesian coordinates of a two-dimensional position vector $\boldsymbol{r}$, and h is a positive real. f has a period D and is assumed to be T times continuously differentiable with T being some non-negative, possibly infinite, integer. A complex scalar field u obeys the Dirichlet condition $u = 0$ on (S) and the scalar Helmholtz equation $\Delta u + k^2 u = 0$ in the upper half-space defined by $z > hf(x)$, with k being a positive real. An incident plane wave ψ_0^- with an implicit time dependence of the type $\mathrm{e}^{-\mathrm{i}\omega t}$ is imposed and u is sought in the form [1]:

$$u(\boldsymbol{r}) = \psi_0^-(\boldsymbol{r}) + u_\mathrm{d}(\boldsymbol{r}), \tag{1}$$

where the diffracted field u_d is equal to a superposition of outgoing and evanescent plane waves (outgoing wave condition [1]), and

$$\psi_n^{\pm}(\boldsymbol{r}) = \exp(\mathrm{i}\boldsymbol{k}_n^{\pm} \cdot \boldsymbol{r}), \tag{2}$$

$$\boldsymbol{k}_n^{\pm} = K_n \boldsymbol{x} \pm k_z(K_n)\boldsymbol{z}, \tag{3}$$