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On new relations in dispersive wave motion

G. Herrmann [a], R. Kienzler [b,*]

[a] Stanford University, Division of Applied Mechanics and Computation, CA, USA
[b] University of Bremen, Department of Production Engineering, Germany

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Abstract

Based on Whitham's variational approach and employing the 4×4 formalism for dispersive wave motion, new balance and conservation laws were established. The general relations are illustrated with a specific example.

Keywords: Dispersive waves; Conservation laws; Balance laws

1. Introduction

Whitham [1,2] has developed a variational approach to study linear and also non-linear wavetrains and its many ramifications and applications in a variety of fields, including modulation theory.

The essence of Whitham's approach consists in postulating a Lagrangian function for the system under consideration, specializing this function for a slowly varying wavetrain, averaging the Lagrangian over one period and, finally, to derive variational equations for this averaged Lagrangian. Since the average variational principle is invariant with respect to a translation in time, the corresponding energy equation was derived, and since it is also invariant to a translation in space, the "wave momentum" equation was also established. Kienzler and Herrmann [3] have shown that the two relations may be derived also by calculating the time rate of change of the average Lagrangian and the spatial gradient of the same function. It is also possible to obtain the energy equation and the three "wave momentum" equations through a simple operation by applying the *grad* operator in four dimensions of space-time. This has been carried out for elastodynamics by Kienzler and Herrmann [4].

The purpose of this contribution is to consider not only the *grad* operator as applied to the average Lagrangian, but additionally also the *div* and *curl* operator, which has not been done before.

* Corresponding author. Tel.: +49 421 218 2255; fax: +49 421 218 7478.
E-mail address: rkienzler@uni-bremen.de (R. Kienzler).

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Section 2 summarizes the basic relations of the problem at hand as presented in Whitham, Kienzler and Herrmann, while Section 3 presents new results stemming from the application of the *div* and *curl* operators. Section 4 concludes with a brief summary and some general comments.

2. Whitham's variational approach

Whitham's variational approach begins with postulating a Lagrangian $L = L\left(\phi, -\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x_i}\right)$ for any system governed by a dependent variable $\phi = \phi(t, x_i)$ where t is the time and x_i are Cartesian coordinates. For linear systems, L is a quadratic function of ϕ and its derivatives.

Next a slowly varying wavetrain is considered

$$\phi \sim a\omega(\theta + \eta), \tag{2.1}$$

where a is the amplitude, η the phase shifting angle and θ is the phase

$$\theta(x_i, t) = x_i k_i - \omega t, \tag{2.2}$$

where k_i is the wave number, $k_i = \frac{\partial \theta}{\partial x_i}$ and ω the frequency, $\omega = -\frac{\partial \theta}{\partial t}$. This form is substituted into the Lagrangian L, derivatives of a, η, ω and k_i are all neglected as being small and the result is averaged over one period

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L \, \mathrm{d}\theta. \tag{2.3}$$

For any linear system, the resulting $\mathcal{L}$ is a function

$$\mathcal{L} = \mathcal{L}(\omega, k_i, a), \tag{2.4}$$

or, more specifically,

$$\mathcal{L} = G(\omega, k_i) a^2, \tag{2.5}$$

where

$$G(\omega, k_i) = 0 \tag{2.6}$$

is the dispersion relation.

Whitham proposes then an "average variational principle"

$$\delta \int \mathcal{L}\left(-\frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x_i}, a\right) \; \mathrm{d}t \, \mathrm{d}x_i = 0 \tag{2.7}$$

for the functions $a(x_i, t)$ and $\theta(x_i, t)$.

Since derivatives of a do not occur, the Euler–Lagrange variational equation for this variable is merely

$$\frac{\partial \mathcal{L}}{\partial a} = 0, \tag{2.8}$$

while the variational equation for θ is

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \theta}{\partial t}\right)}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \theta}{\partial x_i}\right)}\right) = 0, \tag{2.9}$$