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RESEARCH PAPER

Bounding the Efficiency Loss of Mixed Equilibrium in the Transportation Network with Selfish and Altruistic Users

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Abstract: Existing research mainly focuses on the efficiency loss of homogeneous users in the transportation network while little effort has been made to explore the efficiency loss of heterogeneous users. The aim of this article is to investigate the efficiency loss of mixed traffic assignment in the transportation network with selfish and altruistic users. The selfish user chooses a travel path based on the classical user equilibrium (UE) principle and the altruistic user aims to minimize their perceived travel cost (here, the perceived travel cost of each altruistic user is a linear combination of a selfish and altruistic component). Firstly, this article establishes a Variational Inequality (VI) model to depict this mixed traffic assignment. Secondly, the upper bound of this mixed equilibrium traffic assignment is derived by analytic derivation and the relation between the upper bound and the network parameters is obtained. Finally, a numerical example is carried out to validate the analytical result. The analytical and numerical results show that the upper bound of efficiency loss is related to the maximal altruism coefficient, the minimal altruism coefficient and the link travel cost functions.

Key Words: urban traffic; efficiency loss; variational inequality; mixed equilibrium; selfish user; altruistic user

1 Introduction

Recently, quantifying the efficiency loss of heterogeneous user's behavior in the transportation network has been a prevalent topic in traffic science. Roughgarden[1] studied the efficiency loss considering heterogeneous users in network. Liu *et al.*[2] examined the efficiency loss of mixed travel behavior in the transportation network where some users were equipped with the advanced travel information system and others were not. Yu and Huang[3] investigated the upper bound of the efficiency loss of UE-CN mixed equilibrium with polynomial cost functions. Chen and Kempe[4] studied the efficiency loss of a transportation network where all users are altruistic. Karakostas *et al.*[5] analyzed the efficiency loss of the transportation network consisting of selfish and oblivious users (here, the oblivious user always selects the cheapest route in a network without consideration of flow).

This study focuses on the efficiency loss of mixed equilibrium traffic assignment that includes selfish and altruistic users. The selfish user chooses the route based on the classical UE principle and the altruistic user regards the delay that she/he causes for other users in the network. She/he chooses her/his routes to minimize her/his perceived cost which is a linear combination of the selfish and altruistic component. The selfish component is her/his own actual travel cost and the altruistic component is the increment in travel cost that the user causes (precise definitions are provided in Section 2). First, this study formulates the VI model for the selfish-altruistic mixed equilibrium traffic assignment problem. Then, it derives the upper bound of the inefficiency caused by the mixed equilibrium and examines the relationships between the bound and network parameters. Finally, a simple numerical example is given.

2 Model of selfish-altruistic mixed traffic assignment problem

We consider a transportation network $G=(N,A)$ with a finite set of nodes N, and a finite set of directed links A. Let u denote the selfish user and M the set of altruistic users in the network, and each altruistic user $m \in M$ has an altruism coefficient $\beta_m, \beta_m \in [0,1]$. Let W^u be the set of

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all Origin-Destination (OD) pairs whose users obey UE principle. Let W^m be the set of OD pairs whose users are controlled by altruistic user $m \in M$. We supposed that $W^M \equiv \bigcup_{m\in M} W^m$ and $W \equiv W^u \bigcup W^M$. For simplicity and clarity, we assume that the demand d_w between OD pair $w \in W$ is given and fixed . Let $\boldsymbol{d}$ be the vector of demand in the network G. Denote the flow on path $r \in R_W$, $w \in W$, as f_{rW}. $\delta_{ar}^w = 1$ if path $r \in R_w$ traverses link $a \in A$, and 0 otherwise. Let v_a^u denote the flow of link a arising out of the OD pair flows from the set W^u and $\mathbf{v}^u = \left(\cdots, v_{a-1}^u, v_a^u, v_{a+1}^u, \cdots\right)$ denote the vectors of the link flows by selfish user. Let v_a^m be the flow of link a arising out of the OD flows from the set $W^m, m \in M$ and $\mathbf{v}^m \equiv \left(\cdots, v_{a-1}^m, v_a^m, v_{a+1}^m, \cdots\right)$ be the vectors of the link flows by altruistic user m. We define $\boldsymbol{v}^M \equiv \left(\cdots, \boldsymbol{v}^{m-1}, \boldsymbol{v}^m, \boldsymbol{v}^{m+1}, \cdots\right)$ and $\boldsymbol{v} \equiv \left(\boldsymbol{v}^u, \boldsymbol{v}^M\right)$. Denoted by $v_a^M = \sum_{m\in M} v_a^m$ the total flow of altruistic user and $v_a = v_a^u + v_a^M$ is the total flow on link a. The link travel cost function $t_a\left(v_a\right), a \in A$ is separable, differentiable, convex and monotonically increasing with the total link flow v_a. Denoted by $\boldsymbol{t}$, the vector of link travel cost in G.

As shown previously, all the OD demands are fixed. Thus, the feasible sets of link flows of the selfish user and the altruistic users can be defined as follows.

$\Omega^u = \{\boldsymbol{v}^u \mid \boldsymbol{v}^u$ satisfying formulas (1)-(3)$\}$

$$\sum_{r\in R_w} f_{rw} = d_w, \forall w \in W^u \tag{1}$$

$$v_a^u = \sum_{w\in W^u} \sum_{r\in R_w} f_{rw}\delta_{ar}^w,\ \forall a \in A \tag{2}$$

$$f_{rw} \ge 0,\ r \in R_w,\ w \in W^u \tag{3}$$

$\Omega^m = \{\mathbf{v}^m \mid \mathbf{v}^m$ satisfying formulas (4)-(6)$\}$

$$\sum_{r\in R_w} f_{rw} = d_w, \forall w \in W^m \tag{4}$$

$$v_a^m = \sum_{w\in W^m} \sum_{r\in R_w} f_{rw}\delta_{ar}^w,\ \forall a \in A \tag{5}$$

$$f_{rw} \ge 0,\ r \in R_w,\ w \in W^m \tag{6}$$

Let $\Omega = \Omega^u \times \prod_{m\in M} \Omega^m$. Based on the results of Ledyard[6], Chen and Kempe[4], the β altruistic user can be defined as follows.

Definition 1: Each β altruistic user (for $\beta \in [-1,1]$) chooses a path r so as to minimize the cost function

$$t_r^\beta(\mathbf{v}) = (1-\beta)\sum_{a\in A} t_a\left(v_a\right)\delta_{ar} + \beta\sum_{a\in A}\left(t_a\left(v_a\right)v_a\right)'\delta_{ar}$$

$$= (1-\beta)\sum_{a\in r} t_a\left(v_a\right) + \beta\sum_{a\in r}\left(t_a\left(v_a\right)v_a\right)'$$

where, $\sum_{a\in r} t_a\left(v_a\right)$, $\sum_{a\in r}\left(t_a\left(v_a\right)v_a\right)'$ are respectively the selfish part and the altruistic part of the cost. (here $\left(t_a\left(v_a\right)v_a\right)'$ is the derivative with respect to v_a). Note that we can rewrite $t_r^\beta(\boldsymbol{v}) = \sum_{a\in r} t_a\left(v_a\right) + \beta\sum_{a\in r} v_a t_a'\left(v_a\right)$.

The perceived cost of altruistic users equal the sum of the actual travel cost (i.e., selfish component) and the increase in travel cost that the user causes (i.e., the altruistic component). Obviously, $\beta = 0$ means that the users are completely selfish; $\beta = 1$ means that the users are completely altruism; $\beta = -1$ means that the users are completely spiteful. Thus, the perceived cost of the altruistic user m with altruism coefficient β_m on link a can be defined as $t_a^{\beta_m}(\boldsymbol{v}) = t_a\left(v_a\right) + \beta_m v_a t_a'\left(v_a\right), \beta_m \in [0,1], \forall m \in M$.

The aim of selfish users are to minimize the travel cost under the current routing decision of the altruistic users, which is equivalent to solve.

$$\min_{\mathbf{v}^u \in \Omega^u} \sum_{a\in A} \int_0^{v_a^u} t_a\left(v_a^M + x\right)\mathrm{d}x \tag{7}$$

where, the variable v_a^M, $a \in A$ is a fixed constant. If $t_a\left(v_a\right)$ is a strictly increasing function, then the minimization problem in Eq. (7) has a unique solution.

The aim of altruistic user m is to minimize the perceived travel cost of the altruistic user in this specific user under the current routing decision of other users, i.e.,

$$\min_{\mathbf{v}^m \in \Omega^m} \sum_{a\in A} \int_0^{v_a^m} t_a^{\beta_m}\left(v_a^u + v_a^{-m} + x\right)\mathrm{d}x \tag{8}$$

where, $v_a^{-m} = \sum_{i\in M, i\neq m} v_a^i$ and v_a^u, v_a^{-m} are fixed constants. It is easy to conclude that the minimization problem in Eq. (8) has a unique solution by the assumption of $t_a\left(v_a\right)$.

The solution simultaneously satisfies the optimality conditions of the minimization problems in Eq. (7) and Eq. (8) is so-called a selfish-altruistic mixed equilibrium solution. Given the fact that the feasible sets for each user are disjoint, it is well known that the mixed equilibrium can be formulated as the following VI.

Lemma 1 Let $(G, \boldsymbol{d}, \boldsymbol{t}, \beta)$ be a mixed instance consisting of the selfish user and altruistic users. If the separable link travel cost function $t_a\left(v_a\right)$, $a \in A$ is strictly increasing and convex, then the mixed equilibrium of the instance $(G, \boldsymbol{d}, \boldsymbol{t}, \beta)$ is equivalent to finding $\overline{\boldsymbol{v}} \in \Omega$ for each $\boldsymbol{v} \in \Omega$, such that

$$\sum_{a\in A}\left\{t_a\left(\overline{v}_a\right)\left(v_a^u - \overline{v}_a^u\right) + \sum_{m\in M} t_a^{\beta_m}\left(\overline{v}_a\right)\left(v_a^m - \overline{v}_a^m\right)\right\} \ge 0 \tag{9}$$

where, $t_a^{\beta_m}\left(\overline{v}_a\right) = t_a\left(\overline{v}_a\right) + \beta_m \overline{v}_a t_a'\left(\overline{v}_a\right)$.

Since $t_a\left(v_a\right)$ is strictly increasing and convex, the VI problem in Eq. (9) has a solution[7]. Furthermore, the vector of the perceived link costs by the selfish user and altruistic users on link $a \in A$ is $\boldsymbol{c}_a = \left(t_a(v_a), \cdots, t_a(v_a) + \beta_m v_a t'_a(v_a), \cdots\right)$, $m \in M$. If $\boldsymbol{c}_a$ is a strictly monotone function for each link $a \in A$, then the VI problem in Eq. (9) has at most one solution[7].

Let $\overline{\boldsymbol{v}}$ and $\overline{\boldsymbol{v}}_a = \left(\overline{v}_a\right)$, $a \in A$ respectively be the solution vector and the vector of the total link flow of the selfish-altruistic mixed equilibrium under the VI problem in Eq. (9). It is easy to obtain the total travel cost of the system.

$$T(\overline{\boldsymbol{v}}) = \sum_{a\in A} t_a\left(\overline{v}_a\right)\overline{v}_a = \sum_{a\in A} t_a\left(\overline{v}_a\right)\overline{v}_a^u + \sum_{a\in A}\sum_{m\in M} t_a\left(\overline{v}_a\right)\overline{v}_a^m$$