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# Weighted distributions of eigenvalues

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ABSTRACT

In this article, the weighted version of a probability density function is considered as a mapping of the original distribution. Generally, the properties of the distribution of a random matrix and the distributions of its eigenvalues are closely related. Therefore, the weighted versions of the distributions of the eigenvalues of the Wishart distribution are introduced and their properties are discussed. We propose the concept of rotation invariance for the weighted distributions of the eigenvalues of the Wishart and non-central Wishart distributions. We also introduce here, the concept of a "mirror", meaning, looking at the distribution of a random matrix through the distribution of its eigenvalues. Some graphical representations are given, to visualize the weighted distributions of the eigenvalues for specific cases.


## 1. Introduction

Let $X$ be a non-negative random variable (r.v.) having probability density function (p.d.f.) $f(x;\theta)$ where $\theta$ is a scalar or a vector of parameters. Let $w(x) > 0$ be a function

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of $x$, with $E[w(X)] < \infty$. The $w(x)$ is referred to as a weight function. Then, the weighted version of $f(x;\theta)$ is defined as

$$g(x;\theta) = \frac{w(x)f(x;\theta)}{E[w(X)]}. \tag{1}$$

It is possible to let $w(x)$ have a parameter, say, $\psi$, of its own. The p.d.f. $f(x;\theta)$ is referred to as the original distribution and $g(x;\theta)$ is called the weighted version of $f(x;\theta)$. When $w(x) = x$ the p.d.f. $g(.;.)$ is referred to as the length-biased version of the original distribution. It is anticipated that the properties of $g(.;.)$ will closely resemble the properties of $f(.;.)$. Thus, $g(x;\theta)$ can be used for statistical inference about $\theta$. The factor $w(x)/E[w(X)]$ is an operator that changes $f(x;\theta)$ to $g(x;\theta)$. Alternately, the family of $g(.;.)$ can be thought of as a mapping of family of p.d.f.'s $f(.;.)$.

During the past thirty years, a number of papers have appeared discussing univariate weighted/length-biased data and the data analysis using $g(x;\theta)$. In a nutshell, since the eigenvalues explain the most of the variations in the data, the weighted distributions of eigenvalues may have the same importance as the weighted distributions of the random matrix. The importance of distributions of eigenvalues of random matrices is emphasized in literature by [16,15,13,17], amongst others.

There are indications of the presence of size-biased data random matrices, such as example is the biased mutation matrices as refer to in bioinformatics [2]. The bias in the observed values of the eigenvalues is discussed for data collected for face recognition [7]. The presence of bias in signals associated with the length of the signal in MIMO systems are discussed with regard to the eigenvalues of the related matrix configuration [12]. The above examples provide us a necessary motivation for studying the weighted matrix variate distributions and contribute new distributions to matrix theory.

In (1) instead of a univariate r.v. $X$ we can consider a matrix variate r.v. $\boldsymbol{X} : p \times p$ and an associated weight function with related regularity conditions. In this paper, the interest is the derivation of the weighted version of the distribution of eigenvalues and the study of their properties. The stochastic behavior of $\boldsymbol{X}$ is represented by the distribution of its eigenvalues, $\boldsymbol{\lambda}$. Therefore, given the distribution of $\boldsymbol{\lambda}$, certain properties of the distribution of $\boldsymbol{X}$ can be studied. We also introduce a new concept of "mirror" where the p.d.f. $f(.;.)$ asymptotically approaches the p.d.f. $g(.;.)$.

The organization of the paper is as follows: The notation and the definitions needed for the development of this paper are recorded in Section 2. In Section 3, the properties of the length-biased version of the distribution of the eigenvalues of the Wishart matrix are studied and the related results are obtained for the non-central Wishart distribution. Also, is introduced in Section 3 a concept of rotation invariance. In Section 4, a procedure referred to as "mirror", in this paper, is described for reconstructing the original distribution of a random matrix through the weighted version of the distribution of the eigenvalues of the random matrix. To demonstrate the usefulness of this concept of mirror we have provided the numerical results in Section 5. Section 6 concludes with graphical representations of weighted distributions of eigenvalues.