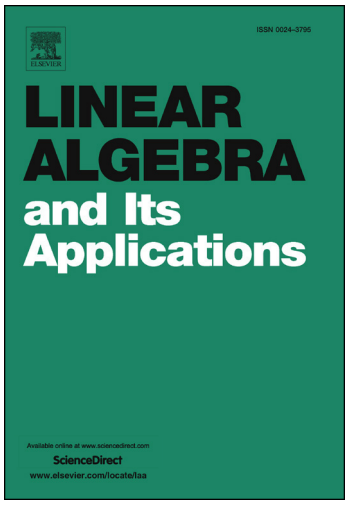

On sequences of Toeplitz matrices over finite fields

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PII: S0024-3795(18)30446-4
DOI: https://doi.org/10.1016/j.laa.2018.09.013
Reference: LAA 14728

To appear in: *Linear Algebra and its Applications*

Received date: 9 April 2018
Accepted date: 10 September 2018

Please cite this article in press as: G. Price, M. Wortham, On sequences of Toeplitz matrices over finite fields, *Linear Algebra Appl.* (2018), https://doi.org/10.1016/j.laa.2018.09.013

ON SEQUENCES OF TOEPLITZ MATRICES OVER FINITE FIELDS

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Abstract. For each non-negative integer n let $\mathcal{A}_n$ be an $n+1$ by $n+1$ Toeplitz matrix over a finite field, F, and suppose for each n that $\mathcal{A}_n$ is embedded in the upper left corner of $\mathcal{A}_{n+1}$. We study the structure of the sequence $\nu = \{\nu_n : n \in \mathbb{Z}^+\}$, where $\nu_n = \text{null}(\mathcal{A}_n)$ is the nullity of $\mathcal{A}_n$. For each $n \in \mathbb{Z}^+$ and each nullity pattern $\nu_0, \nu_1, \ldots, \nu_n$, we count the number of strings of Toeplitz matrices $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ with this pattern. As an application we present an elementary proof of a result of D. E. Daykin on the number of $n \times n$ Toeplitz matrices over $GF(2)$ of any specified rank.
2000 MSC Classification 15A33, 15A57
Keywords: Toeplitz matrix, nullity sequence, rank, finite fields

1. Introduction

In this paper we consider sequences $\{\mathcal{A}_n\}$ of Toeplitz matrices with entries in an arbitrary finite field F. The sequence $\{\mathcal{A}_n\}$ is uniquely determined by a pair of sequences $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ over F, so that $\mathcal{A}_n$ is the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} & a_n \\ b_1 & a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\ b_2 & b_1 & a_0 & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_n & b_{n-1} & b_{n-2} & \ldots & b_1 & a_0 \end{bmatrix} \tag{1.1}$$

In particular, $\mathcal{A}_n$ lies in the top left corner (as well as the bottom right corner) of $\mathcal{A}_{n+1}$.

For each sequence $\{\mathcal{A}_n\}$ let $\nu = \{\nu_n\}$ be the corresponding sequence of nullities, i.e. ν_n is the nullity of $\mathcal{A}_n$. As in [1], where K. Culler and one of the authors considered nullity sequences of skew centro-symmetric matrices $\mathcal{A}_n$ (see also [10],[11]), here we ascertain the patterns of the nullity sequences which can occur for Toeplitz matrices. We present an elementary proof which shows that $\{\nu_n\}$ is a concatenation of strings of the form $0, 0, \ldots, 0$ or $1, 2, \ldots, d-1, d, d, \ldots, d, d-1, d-2, \ldots, 1, 0$, where d can repeat any finite number of times; or the nullity sequence may consist of a concatenation of finitely many strings of this type, followed by $1, 2, \ldots$, see Theorem 1 . These patterns were observed in [4] for sequences of Toeplitz matrices over $\mathbb{C}$. We analyze the structure of the (right) kernels of matrices which satisfy these patterns and use this to determine the number of matrices $\mathcal{A}_0, \ldots, \mathcal{A}_n$ which satisfy a specific nullity pattern, $\nu_0, \ldots, \nu_n$. We note that the structure of kernels of Toeplitz matrices over $\mathbb{C}$ is carried out in [4], and a similar analysis is carried

Date: 13 August 2018.
* supported in part by the United States Naval Research Laboratory, Washington, DC.