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On rank range of interval matrices

Elena Rubei

Dipartimento di Matematica e Informatica "U. Dini", viale Morgagni 67/A, 50134 Firenze, Italy

A R T I C L E I N F O

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A B S T R A C T

An interval matrix is a matrix whose entries are intervals in $\mathbb{R}$. Let $p, q \in \mathbb{N} \setminus \{0\}$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix; given a $p \times q$ matrix A with entries in $\mathbb{R}$, we say that $A \in \boldsymbol{\alpha}$ if $a_{i,j} \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ for any i, j. We establish a criterion to say if an interval matrix contains a matrix of rank 1. Moreover we determine the maximum rank of the matrices contained in a given interval matrix. Finally, for any interval matrix $\boldsymbol{\alpha}$ with no more than 3 columns, we describe a way to find the range of the ranks of the matrices contained in $\boldsymbol{\alpha}$.

1. Introduction

Let $p, q \in \mathbb{N} \setminus \{0\}$; a $p \times q$ interval matrix is a $p \times q$ matrix whose entries are intervals in $\mathbb{R}$; let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix, where $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}$ are real numbers with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i and j; given a $p \times q$ matrix A with entries in $\mathbb{R}$, we say that $A \in \boldsymbol{\alpha}$ if $a_{i,j} \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ for any i, j. In this paper we investigate about the range of the ranks of the matrices contained in $\boldsymbol{\alpha}$.

There are several papers studying when an interval $p \times q$ matrix $\boldsymbol{\alpha}$ has full rank, that is when all the matrices contained in $\boldsymbol{\alpha}$ have rank equal to $\min\{p, q\}$. For any $p \times q$ interval

E-mail address: elena.rubei@unifi.it.

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matrix $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$, let $\mathrm{mid}(\boldsymbol{\alpha})$, $\mathrm{rad}(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}|$ be respectively the midpoint, the radius and the modulus of $\boldsymbol{\alpha}$, that is the $p \times q$ matrices such that

$$\mathrm{mid}(\boldsymbol{\alpha})_{i,j} = \frac{\underline{\alpha}_{i,j} + \overline{\alpha}_{i,j}}{2}, \qquad \mathrm{rad}(\boldsymbol{\alpha})_{i,j} = \frac{\overline{\alpha}_{i,j} - \underline{\alpha}_{i,j}}{2}, \qquad |\boldsymbol{\alpha}|_{i,j} = \max\{|\underline{\alpha}_{i,j}|, |\overline{\alpha}_{i,j}|\}$$

for any i, j. The following theorem characterizes full-rank square interval matrices:

Theorem 1 *(Rohn, [7]). Let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times p$ interval matrix, where $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i, j. Let $Y_p = \{-1, 1\}^p$ and, for any $x \in Y_p$, denote by T_x the diagonal matrix whose diagonal is x.*

Then $\boldsymbol{\alpha}$ is a full-rank interval matrix if and only if, for each $x, y \in Y_p$,

$$\det\Big(\mathrm{mid}(\boldsymbol{\alpha})\Big)\,\det\Big(\mathrm{mid}(\boldsymbol{\alpha}) - T_x\,\mathrm{rad}(\boldsymbol{\alpha})\,T_y\Big) > 0.$$

See [7] and [8] for other characterizations. The following theorem characterizes full-rank $p \times q$ interval matrices, see [9], [10], [12]:

Theorem 2 *(Rohn). A $p \times q$ interval matrix $\boldsymbol{\alpha}$ with $p \geq q$ has full rank if and only if the system of inequalities*

$$|\mathrm{mid}(\boldsymbol{\alpha})\,x| \leq \mathrm{rad}(\boldsymbol{\alpha})\,|x|, \qquad x \in \mathbb{R}^q$$

has only the trivial solution $x = 0$.

A problem which can be connected with the quoted ones is the one of the partial matrices: let K be a field; a partial matrix over K is a matrix where only some of the entries are given and they are elements of K; a completion of a partial matrix is a specification of the unspecified entries. We say that a submatrix of a partial matrix is specified if all its entries are given. The problem of determining whether, given a partial matrix, a completion with some prescribed property exists and related problems have been widely studied: we quote, for instance, the papers [1], [2], [14]. In particular there is a wide literature about the problem of determining the maximal and the minimal rank of the completions of a partial matrix.

In [2], Cohen, Johnson, Rodman and Woerdeman determined the maximal rank of the completions of a partial matrix in terms of the ranks and the sizes of its maximal specified submatrices; see also [1] for the proof. The problem of determining the minimal rank of the completions of a partial matrix seems more difficult and it has been solved only in some particular cases, see for instance the paper [13] for the case of triangular matrices. We quote also the paper [3], where the authors estabish a criterion to say if a partial matrix has a completion of rank 1.

Also for interval matrices, the problem of determining the miminal rank of the matrices contained in a given interval matrix seems much more difficult than the problem of