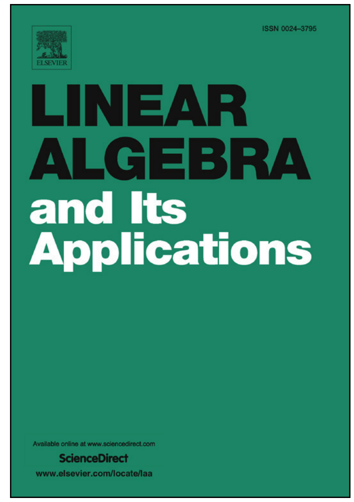

The Schwarz inequality via operator-valued inner product and the geometric operator mean

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PII: S0024-3795(18)30466-X
DOI: https://doi.org/10.1016/j.laa.2018.09.025
Reference: LAA 14740

To appear in: *Linear Algebra and its Applications*

Received date: 6 May 2018
Accepted date: 22 September 2018

THE SCHWARZ INEQUALITY VIA OPERATOR-VALUED INNER PRODUCT AND THE GEOMETRIC OPERATOR MEAN

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Abstract. In this paper, by virtue of the Cauchy-Schwarz operator inequality due to J.I. Fujii, we show weighted mixed Schwarz operator inequalities in terms of the geometric operator mean and its Lin's type refinement. As applications, we show Wielandt type operator inequalities that refine the weighted mixed Schwarz operator inequality under some orthogonal conditions. Moreover, we show the variance-covariance operator inequality via the geometric operator mean which differs from Bhatia-Davis's one and estimate the upper bounds. By our formulation, we show a Robertson type inequality associated to a unital completely positive linear map on $B(H)$.

1. Introduction

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis. Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For selfadjoint operators A and B, the order relation $A \geq B$ means that $A - B$ is positive. Regarding a sesquilinear map $B(X,Y) = Y^*X$ for $X, Y \in B(H)$ as an operator-valued inner product, several operator versions for the Schwarz inequality are discussed by many researchers. For example, if $X, Y \in B(H)$, then the Schwarz inequality for operators

$$X^*Y(Y^*Y)^{-1}Y^*X \leq X^*X \tag{1.1}$$

holds. Indeed, since $Y(Y^*Y + \varepsilon I)^{-1}Y^* \leq I$ for all $\varepsilon > 0$, there exists the strong operator limit of $Y(Y^*Y + \varepsilon I)^{-1}Y^*$ as $\varepsilon \to 0$ and we define

$$Y(Y^*Y)^{-1}Y^* = \operatorname*{s-lim}_{\varepsilon \to 0} Y(Y^*Y + \varepsilon I)^{-1}Y^*$$

and write $Y(Y^*Y)^{-1}Y^* \in B(H)$. This formulation for matrices is firstly given by Marshall and Olkin in [15]. Let T be a positive operator and X, Y any two operators in $B(H)$. Replacing X and Y in (1.1) by $T^{1/2}X$ and $T^{1/2}Y$, respectively, we obtain $X^*TY(Y^*TY)^{-1}Y^*TX \in B(H)$ and

$$X^*TY(Y^*TY)^{-1}Y^*TX \leq X^*TX. \tag{1.2}$$

In [3], Bhatia and Davis showed some new operator versions of the Schwarz inequality for a positive linear map, which is a generalization of (1.2): A map Φ on $B(H)$ is called 2-positive if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0 \qquad \text{implies} \qquad \begin{pmatrix} \Phi(A) & \Phi(B) \\ \Phi(C) & \Phi(D) \end{pmatrix} \geq 0.$$

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A64.

Key words and phrases. Cauchy-Schwarz inequality, geometric operator mean, Wielandt inequality, variance-covariance inequality.