

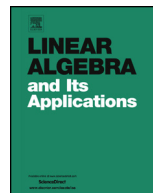


ELSEVIER

Contents lists available at ScienceDirect

# Linear Algebra and its Applications

[www.elsevier.com/locate/laa](http://www.elsevier.com/locate/laa)



## Linear preservers for the $q$ -permanent, cycle $q$ -permanent expansions, and positive crossings in digraphs <sup>☆</sup>

Eduardo Marques de Sá

*CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal*

### ARTICLE INFO

#### Article history:

Received 22 March 2018

Accepted 27 September 2018

Available online 28 September 2018

Submitted by R. Brualdi

#### MSC:

15A15

15A86

05C20

05C50

#### Keywords:

$q$ -Permanent

Determinant

Polynomial identities

Digraphs

Permutations

### ABSTRACT

The  $q$ -permanent linear preservers are described. We give several expansion formulas for the  $q$ -permanent of a square matrix, based on the cycle factorization of permutations. Some of these formulas are valid for all matrices, but others are not; for each such formula  $\Phi$  we determine all digraphs  $D$  such that  $\Phi$  holds for all matrices with digraph  $D$ . Proof techniques are based on combinatorial results, relating the length (number of inversions) of a permutation, the lengths of its cycles, and a delicate counting of crossings, jumps, and arc-under-arc relations in digraphs. We get new algebraic characterizations of noncrossing [acyclic] graphs.

© 2018 Elsevier Inc. All rights reserved.

<sup>☆</sup> Work partially supported by the Centre for Mathematics of the University of Coimbra, UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC, and co-funded by the European Regional Development Fund, through the Partnership Agreement PT2020.

*E-mail address:* [emsa@mat.uc.pt](mailto:emsa@mat.uc.pt).

## 1. Introduction

This paper is a continuation of [19], on the  $q$ -permanent of an  $n$ -square matrix  $A = (a_{ij})$ , a polynomial given by

$$\text{per}_q A = \sum_{\sigma \in \mathcal{S}_n} q^{\ell(\sigma)} \prod_{i=1}^n a_{i\sigma_i}.$$

Here,  $\mathcal{S}_n$  is the symmetric group of order  $n$ , and  $\ell(\sigma)$  denotes the *length* of  $\sigma$ , defined as the number of inversions of the permutation  $\sigma$ . In [5,14,21,25] the reader will find the genesis and uses of this function in the areas of mathematical physics, and quantum groups and algebras. Further developments may be found in [1,2,11,12].

Section 3 describes the  $q$ -permanent linear preservers. In [22,23] the  $q$ -permanent is generalized to multivariable quantum parameters and, in this context, some expansions are obtained for the  $q$ -permanent which are reminiscent of the archetypal expansions of Laplace along a set of rows or columns. The expansions considered below (in sections 4, 7, and 8) are of a different nature, in that we collect the  $q$ -permanent terms according to the cycle structure of the digraph of the matrix  $A$ , as has been done for the determinant in [16–18]. In section 5, we relate the length of a permutation, the lengths of its cycles, and the number of positive crossings in the corresponding digraphs. In section 6 we show how to express the number of positive crossings, using jumps of arcs over vertices, and arc-under-arc relations. This paves the way to sections 7 and 8, where the main  $q$ -permanent expansion formula is modified in several ways. Each modified formula  $\Phi$  is *combinatorially solved*, *i.e.*, all digraphs  $D$  are found such that  $\Phi$  holds for the generic matrices with digraph  $D$ .

In the referred combinatorial solutions to our tentative expansions of the  $q$ -permanent, new algebraic characterizations emerge for interesting classes of graphs, like noncrossing graphs, and noncrossing acyclic graphs.

## 2. Preliminaries

On digraphs, graphs and matrices we follow the traditional concepts as may be seen in, *e.g.*, [6,7], with minor changes. The set  $V(D)$  of the vertices of a [di]graph  $D$  is a subset of  $[n] = \{1, \dots, n\}$ . Notations like  $(i, j) \in E \subseteq D$  mean that  $(i, j)$  is an arc of  $E$ , and  $E$  is a subdigraph of  $D$ ; an arc is also denoted  $i \rightarrow j$ . We write  $[r, s]$ ,  $]r, s]$ , etc., to refer *integer intervals*. By *disjoint digraphs* we mean *vertex disjoint digraphs*, unless otherwise specified.

On the concept of (*oriented*) *cycle*, as a digraph and as a permutation, we follow the conventions of [19], except that a loop is considered here as a cycle. Thus a  $k$ -cycle, often denoted in short notation,  $c = (v_1 v_2 \cdots v_k)$ , is a digraph with vertex set  $V(c) = \{v_1, \dots, v_k\}$ , and arcs  $v_i \rightarrow v_{i+1}$ , with  $i$  read modulo  $k$ . The set of all cycles through a given vertex  $v$  is denoted by  $\mathcal{C}_v$ , or  $\mathcal{C}_v(n)$  if needed; we may identify  $\mathcal{C}_v$  with a set of cyclic permutations of  $\mathcal{S}_n$ . The sole 1-cycle of  $\mathcal{C}_v$  is the loop  $(v)$ .

Download English Version:

<https://daneshyari.com/en/article/10997870>

Download Persian Version:

<https://daneshyari.com/article/10997870>

[Daneshyari.com](https://daneshyari.com)