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Linear preservers for the q-permanent, cycle q-permanent expansions, and positive crossings in digraphs ☆

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The q-permanent linear preservers are described. We give several expansion formulas for the q-permanent of a square matrix, based on the cycle factorization of permutations. Some of these formulas are valid for all matrices, but others are not; for each such formula Φ we determine all digraphs D such that Φ holds for all matrices with digraph D. Proof techniques are based on combinatorial results, relating the length (number of inversions) of a permutation, the lengths of its cycles, and a delicate counting of crossings, jumps, and arc-under-arc relations in digraphs. We get new algebraic characterizations of noncrossing [acyclic] graphs.

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1. Introduction

This paper is a continuation of [19], on the q-permanent of an n-square matrix $A = (a_{ij})$, a polynomial given by

$$\operatorname{per}_q A = \sum_{\sigma \in \mathscr{S}_n} q^{\ell(\sigma)} \prod_{i=1}^n a_{i\sigma_i} \,.$$

Here, $\mathscr{S}_n$ is the symmetric group of order n, and $\ell(\sigma)$ denotes the *length* of σ, defined as the number of inversions of the permutation σ. In [5,14,21,25] the reader will find the genesis and uses of this function in the areas of mathematical physics, and quantum groups and algebras. Further developments may be found in [1,2,11,12].

Section 3 describes the q-permanent linear preservers. In [22,23] the q-permanent is generalized to multivariable quantum parameters and, in this context, some expansions are obtained for the q-permanent which are reminiscent of the archetypal expansions of Laplace along a set of rows or columns. The expansions considered below (in sections 4, 7, and 8) are of a different nature, in that we collect the q-permanent terms according to the cycle structure of the digraph of the matrix A, as has been done for the determinant in [16–18]. In section 5, we relate the length of a permutation, the lengths of its cycles, and the number of positive crossings in the corresponding digraphs. In section 6 we show how to express the number of positive crossings, using jumps of arcs over vertices, and arc-under-arc relations. This paves the way to sections 7 and 8, where the main q-permanent expansion formula is modified in several ways. Each modified formula Φ is *combinatorially solved*, *i.e.*, all digraphs D are found such that Φ holds for the generic matrices with digraph D.

In the referred combinatorial solutions to our tentative expansions of the q-permanent, new algebraic characterizations emerge for interesting classes of graphs, like noncrossing graphs, and noncrossing acyclic graphs.

2. Preliminaries

On digraphs, graphs and matrices we follow the traditional concepts as may be seen in, *e.g.*, [6,7], with minor changes. The set $V(D)$ of the vertices of a [di]graph D is a subset of $[n] = \{1, \dots, n\}$. Notations like $(i, j) \in E \subseteq D$ mean that (i, j) is an arc of E, and E is a subdigraph of D; an arc is also denoted $i \to j$. We write $[r, s]$, $]r, s]$, etc., to refer *integer intervals*. By *disjoint digraphs* we mean *vertex* disjoint digraphs, unless otherwise specified.

On the concept of *(oriented) cycle*, as a digraph and as a permutation, we follow the conventions of [19], except that a loop is considered here as a cycle. Thus a k-cycle, often denoted in short notation, $c = (v_1 v_2 \cdots v_k)$, is a digraph with vertex set $V(c) = \{v_1, \dots, v_k\}$, and arcs $v_i \to v_{i+1}$, with i read modulo k. The set of all cycles through a given vertex v is denoted by $\mathscr{C}_v$, or $\mathscr{C}_v(n)$ if needed; we may identify $\mathscr{C}_v$ with a set of cyclic permutations of $\mathscr{S}_n$. The sole 1-cycle of $\mathscr{C}_v$ is the loop (v).