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Locality of connective constants

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ABSTRACT

The connective constant $\mu(G)$ of a quasi-transitive graph G is the exponential growth rate of the number of self-avoiding walks from a given origin. We prove a locality theorem for connective constants, namely, that the connective constants of two graphs are close in value whenever the graphs agree on a large ball around the origin (and a further condition is satisfied). The proof is based on a generalized bridge decomposition of self-avoiding walks, which is valid subject to the assumption that the underlying graph is quasi-transitive and possesses a so-called *unimodular graph height function*.

1. Introduction, and summary of results

There is a rich theory of interacting systems on infinite graphs. The probability measure governing a process has, typically, a continuously varying parameter, z say, and there is a singularity at some 'critical point' z_c. The numerical value of z_c depends in general on the choice of underlying graph G, and a significant part of the associated literature is directed towards estimates of z_c for different graphs. In most cases of interest, the value of z_c depends on more than the geometry of some bounded domain only. This observation provokes the question of 'locality': to what degree is the value of z_c determined by the knowledge of a bounded domain of G?

The purpose of the current paper is to present a locality theorem (namely, Theorem 5.1) for the connective constant $\mu(G)$ of the graph G. A self-avoiding walk (SAW) is a path that visits no vertex more than once. SAWs were introduced in the study of long-chain polymers in chemistry (see, for example, the 1953 volume of Flory, [14]), and their theory has been much developed since (see the book of Madras and Slade, [28], and the recent review [2]). If the underlying graph G has some periodicity, the number of n-step SAWs from a given origin grows exponentially with some growth rate $\mu(G)$ called the *connective constant* of the graph G.

There are only few graphs G for which the numerical value of $\mu(G)$ is known exactly (detailed references for a number of such cases may be found in [17]), and a substantial part of the literature on SAWs is devoted to inequalities for $\mu(G)$. The current paper may be viewed in this light, as a continuation of the series of papers on the broad topic of connective constants of transitive graphs by the same authors, see [15–17,20].

The main result (Theorem 5.1) of this paper is as follows. Let G, G' be infinite, vertex-transitive graphs, and write $S_K(v, G)$ for the K-ball around the vertex v in G. If $S_K(v, G)$ and $S_K(v', G')$ are isomorphic as rooted graphs, then

$$|\mu(G) - \mu(G')| \le \epsilon_K(G), \tag{1.1}$$

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where $\epsilon_K(G) \to 0$ as $K \to \infty$. (A related result holds for quasi-transitive graphs.) This is proved subject to certain conditions on the graphs G, G', of which the primary condition is that they support so-called 'unimodular graph height functions' (see Section 3 for the definition of a graph height function). The existence of unimodular graph height functions permits the use of a 'bridge decomposition' of SAWs (in the style of the work of Hammersley and Welsh [24]), and this leads in turn to computable sequences that converge to $\mu(G)$ from above and below, respectively. The locality result of (1.1) may be viewed as a partial answer to a question of Benjamini, [3, Conj. 2.3], made independently of the work reported here.

A class of vertex-transitive graphs of special interest is provided by the Cayley graphs of finitely generated groups. Cayley graphs have algebraic as well as geometric structure, and this allows a deeper investigation of locality and of graph height functions. The corresponding investigation is reported in the companion paper [18] where, in particular, we present a method for the construction of a graph height function via a suitable harmonic function on the graph.

The locality question for percolation was approached by Benjamini, Nachmias, and Peres [4] for tree-like graphs. Let G be vertex-transitive with degree $d + 1$. It is elementary that the percolation critical point satisfies $p_c \geq 1/d$ (see [7, Thm 7]), and an asymptotically equivalent upper bound for p_c was developed in [4] for a certain family of graphs which are (in a certain sense) locally tree-like. In recent work of Martineau and Tassion [29], a locality result has been proved for percolation on abelian graphs. The proof extends the methods and conclusions of [21], where it is proved that the slab critical points converge to $p_c(\mathbb{Z}^d)$, in the limit as the slabs become infinitely 'fat'. (A related result for connective constants is included here at Example 5.3.)

We are unaware of a locality theorem for the critical temperature T_c of the Ising model. Of the potentially relevant work on the Ising model to date, we mention [6,8,10,11,26,31].

This paper is organized as follows. Relevant background and notation is described in Section 2. The concept of a graph height function is presented in Section 3, where examples are included of infinite graphs with graph height functions. Bridges and the bridge constant are defined in Section 4, and it is proved in Theorem 4.3 that the bridge constant equals the connective constant whenever there exists a unimodular graph height function. The main 'locality theorem' is given at Theorem 5.1. Theorem 5.2 is an application of the locality theorem in the context of a sequence of quotient graphs; this parallels the Grimmett–Marstrand theorem [21] for percolation on (periodic) slabs, but with the underlying lattice replaced by a transitive graph with a unimodular graph height function. Sections 6 and 7 contain the proofs of Theorem 4.3.

2. Notation and background

The graphs $G = (V, E)$ considered here are generally assumed to be infinite, connected, locally finite, undirected, and also simple, in that they have neither loops nor multiple edges. An edge e with endpoints u, v is written as $e = \langle u, v\rangle$. If $\langle u, v\rangle \in E$, we call u and v *adjacent*, and we write $u \sim v$. The set of neighbors of v is written as $\partial v = \{u \in V : \langle u, v\rangle \in E\}$.

Loops and multiple edges have been excluded for cosmetic reasons only. A SAW can traverse no loop, and thus loops may be removed without changing the connective constant. The same proofs are valid in the presence of multiple edges. When there are multiple edges, we are effectively considering SAWs on a weighted simple graph, and indeed our results are valid for edge-weighted graphs with strictly positive weights, and for counts of SAWs in which the contribution of a given SAW is the product of the weights of the edges therein.

The *degree* of vertex v is the number of edges incident to v, denoted $\deg_G(v)$ or $\deg(v)$, and $G = (V, E)$ is called *locally finite* if every vertex-degree is finite. The maximum vertex-degree is denoted $\delta_G = \sup\{\deg_G(v) : v \in V\}$. The *graph-distance* between two vertices u, v is the number of edges in the shortest path from u to v, denoted $d_G(u, v)$. We denote by $S_k(v) = S_k(v, G)$ the ball of G with center v and radius k.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. A subgroup $\Gamma \leq \mathrm{Aut}(G)$ is said to *act transitively* on G (or on its vertex-set V) if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there is a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *(vertex-)transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively). For a subgroup $\mathcal{H} \leq \mathrm{Aut}(G)$ and a vertex $v \in V$, the orbit of v under $\mathcal{H}$ is written $\mathcal{H}v$. The number of such orbits is written as $M(\mathcal{H}) = |G/\mathcal{H}|$.

A *walk* w on G is an (ordered) alternating sequence $(w_0, e_0, w_1, e_1, \dots, e_{n-1}, w_n)$ of vertices w_i and edges $e_i = \langle w_i, w_{i+1}\rangle$, with $n \geq 0$. We write $|w| = n$ for the *length* of w, that is, the number of edges in w. The walk w is called *closed* if $w_0 = w_n$. We note that w is directed from w_0 to w_n. When, as generally assumed, G is simple, we may abbreviate w to the sequence $(w_0, w_1, \dots, w_n)$ of vertices visited.

A *cycle* is a closed walk w traversing three or more distinct edges, and satisfying $w_i \neq w_j$ for $1 \leq i < j \leq n$. Strictly speaking, cycles (thus defined) have orientations derived from the underlying walk, and for this reason we may refer to them sometimes as *directed* cycles of G.

An *n-step self-avoiding walk* (SAW) on G is a walk containing n edges no vertex of which appears more than once. Let $\Sigma_n(v)$ be the set of n-step SAWs starting at v, with cardinality $\sigma_n(v) := |\Sigma_n(v)|$, and let

$$\sigma_n = \sigma_n(G) := \sup\{\sigma_n(v) : v \in V\}. \tag{2.1}$$

We have in the usual way (see [23,28]) that

$$\sigma_{m+n} \leq \sigma_m \sigma_n, \tag{2.2}$$