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Odd connection and odd vertex-connection of graphs

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ABSTRACT

In the last years, connection concepts such as rainbow connection and proper connection appeared in graph theory and received a lot of attention. In this paper, we present a general concept of connection in graphs. As a particular case, we introduce the odd connection number and the odd vertex-connection number of a graph. Furthermore, we compute and study the odd connection number and the odd vertex-connection number of graphs of various graph classes.

1. Introduction

In this paper, we consider simple and undirected graphs only. For notion and graph theoretic terminology, we generally follow the book of West [14]. Aside from that, we denote by $n(G) := |V(G)|$ the order of a graph G, by $n_i(G)$ the number of degree i vertices of G for $i \in \mathbb{N}$, and by $m(G) := |E(G)|$ the size of G. Furthermore, for simplicity, we denote by $[k]$ the set $\{1, 2, \dots, k\}$.

Let G be a graph. A *path* P in G is a subgraph of G consisting of a vertex set $\{v_1, v_2, \dots, v_k\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{k-1}v_k\}$. For simplicity, we write $P: v_1v_2 \dots v_k$ and, if P contains two vertices u, v, then we denote by uPv the subpath of P between u and v. Two vertices, say $u, v \in V(G)$, are *connected* by a path P if the end vertices of P are u and v. We note that, since we consider undirected graphs, we do not care about the direction of a path. In particular, if P is a path and $u, v \in V(P)$, then $uPv = vPu$.

Let $\mathbb{A}$ be a finite alphabet, e.g. a set of colours, digits, symbols, ..., whose elements are called *letters*. A *word* W of length k over $\mathbb{A}$ is a sequence of letters, say $W: a_1a_2a_3 \dots a_k$ where $a_i \in \mathbb{A}$ for all $i \in [k]$. Let us recall some properties that words can have (see e.g. [5]). A word W is

... *proper* if consecutive letters are not identical,
... *rainbow* if it does not contain two identical letters,
... *conflict-free* if at least one letter occurs exactly once in W,
... *monochromatic* if all letters are identical,
... *odd* if each letter of the alphabet $\mathbb{A}$ appears an odd number of times or zero times in W.

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Using properties of words over an alphabet $\mathbb{A}$, we can introduce two graph theoretic meta-concepts as follows:

Consider a graph G. If there is an edge colouring $c_E: E(G) \to \mathbb{A}$, then we can associate a path $P: v_1v_2 \dots v_k$ with $v_i \in V(G)$ for all $i \in [k]$, P is necessarily a subgraph of G, with a word $c_E(v_1v_2)c_E(v_2v_3)\dots c_E(v_{k-1}v_k)$. Similarly, if $c_V: V(G) \to \mathbb{A}$ is a vertex colouring, then we can associate P with the word $c_V(v_1)c_V(v_2)\dots c_V(v_k)$. Now, let $\mathcal{P}$ be a property for a word over the alphabet $\mathbb{A}$. Considering an edge colouring $c_E: E(G) \to \mathbb{A}$ or a vertex colouring $c_V: V(G) \to \mathbb{A}$, we say that P *has property* $\mathcal{P}$ if the associated word $c_E(v_1v_2)c_E(v_2v_3)\dots c_E(v_{k-1}v_k)$ or $c_V(v_1)c_V(v_2)\dots c_V(v_k)$ has property $\mathcal{P}$, respectively.

Let G be a connected graph, $\mathbb{A}$ be an alphabet with k letters, $\mathcal{P}$ be a property for words, and $c_E: E(G) \to \mathbb{A}$ be an edge colouring ($c_V: V(G) \to \mathbb{A}$ be a vertex colouring). The edge colouring c_E *makes* G $\mathcal{P}$ *connected* (the vertex colouring c_V *makes* G $\mathcal{P}$ *vertex-connected*) if every two distinct vertices of G are connected by a path having property $\mathcal{P}$. The minimum integer k for which there exists an edge colouring $c_E: E(G) \to \mathbb{A}$ (a vertex colouring $c_V: V(G) \to \mathbb{A}$) with $|\mathbb{A}| = k$ that makes G $\mathcal{P}$ connected ($\mathcal{P}$ vertex-connected) is the $\mathcal{P}$ *connection number* ($\mathcal{P}$ *vertex-connection number*) of G.

From a practical point of view, $\mathcal{P}$ connection of graphs plays an important role for security and accessibility in communication networks. While the information which one sends through a network from one node to another has to be protected by passwords, it is of high importance that the password sequences of information transferring paths meet some prescribed requirements. Since managing a whole bunch of passwords, that are assigned to direct information transferring paths between two nodes, is expensive, it is a natural question to ask for a minimum number of passwords securing the information transferring paths. By representing each node as a vertex, each direct information transferring path between two nodes as an edge, each possible password by a different colour, and the prescribed requirements for the password sequences of information transferring paths by a property $\mathcal{P}$ for a word over the alphabet of colours, the above problem is translated to finding the $\mathcal{P}$ connection number of a path.

For example, after 11 September 2001, Ericksen made the following observation [7]:

> "*An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning there was no way for officers and agents to cross check information between various organizations.*"

As an immediate idea solving this deficiency, Chartrand et al. introduced the concept of rainbow connection in [4]; that is, the authors have introduced the concept of $\mathcal{P}$ connection where property $\mathcal{P}$ is rainbow.

Similarly, an application for $\mathcal{P}$ vertex-connection can be described as follows. One can send information through a network from one node to another one, in each node on the information transferring path, the information is processed, and one can ask that the sequence of processing steps for an information meets some prescribed requirements. To be more precise, it is of high importance to minimise the number of different processing steps for an assignment of processing steps to the nodes of the network under the assumption that between every two nodes there exists some information transferring path whose password sequence meets the prescribed requirements. The problem can be translated to finding the $\mathcal{P}$ vertex-connection number in a graph G whose vertices represent the nodes and whose edges the direct information transferring paths between two nodes in the network. We ask for the $\mathcal{P}$ vertex-connection number of G, where colours model processing steps and the prescribed requirements for the information transferring paths are a property $\mathcal{P}$ a word can have over the alphabet of colours.

Returning to the mathematical problem and its applications, the $\mathcal{P}$ connection number and the $\mathcal{P}$ vertex-connection number of graphs have been intensively studied for some of the mentioned properties, for example if property $\mathcal{P}$ is rainbow [3,4,8–11], proper [1,13], or conflict-free [6]. In this paper, we concentrate on odd connection and odd vertex-connection of simple connected graphs.

We say that a path P, which is a subgraph of a graph G equipped with an edge colouring $c_E: E(G) \to [k]$ (a vertex colouring $c_V: V(G) \to [k]$), is *odd coloured* if each colour of $[k]$ is used either an odd or zero number of times for the edges (vertices) of P.

A connected graph G, coloured by an edge colouring $c_E: E(G) \to [k]$ (a vertex colouring $c_V: V(G) \to [k]$), is said to be *odd connected* (*odd vertex-connected*) if every two distinct vertices are connected by an odd coloured path. The least integer k, for which we have an edge colouring $c_E: E(G) \to [k]$ (a vertex colouring $c_V: E(G) \to [k]$) that makes G odd connected (odd vertex-connected), is called the *odd connection number* (*odd vertex-connection number*) of G, denoted by $\mathrm{oc}(G)$ (by $\mathrm{ovc}(G)$).

Let $\ell, k \geq 1$ be two integers such that $\ell \leq k$. We say that an edge colouring $c_E: E(G) \to [k]$ (a vertex colouring $c_V: V(G) \to [k]$) makes G *ℓ-switchable odd connected* (*ℓ-switchable odd vertex-connected*) if, for every two distinct vertices, say u and v, there exist two not necessarily vertex or edge disjoint paths, say P^1 and P^2, connecting u and v in G such that all edges (vertices) of P^1 are coloured by c_E (by c_V) with colour ℓ, and the number, say p_2, of edges (vertices) of P^2 which are coloured by c_E (by c_V) with colour ℓ satisfies $p_2 \in \{m(P^2) - 1, m(P^2)\}$ and $m(P^1) - p_2 = 1 \bmod 2$ (satisfies $p_2 \in \{n(P^2) - 1, n(P^2)\}$ and $n(P^1) - p_2 = 1 \bmod 2$.) In other words, c_E (c_V) makes G ℓ-switchable odd connected (ℓ-switchable odd vertex-connected) if and only if, for every two distinct vertices $u, v \in V(G)$, there exist two paths P^1 and P^2 connecting u and v in G such that either the edges (vertices) of P^1 and P^2 are coloured by c_E (by c_V) with colour ℓ and the lengths of the two paths P^1, P^2 are of different parity, or all edges (vertices) of P^1 and all but one edges (vertices) of P^2 are coloured by c_E (by c_V) with colour ℓ and the lengths of the two paths P^1, P^2 are of the same parity. Furthermore, we note that if c_E (c_V) makes G ℓ-switchable odd connected (ℓ-switchable odd vertex-connected), then c_E (c_V) makes G odd connected (odd vertex-connected).