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On the use of the phase closure principle to calculate the natural frequencies of a rod or beam with nonlinear boundaries

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The calculation of natural frequencies of rods and beams with stiffness boundaries is a common problem encountered by vibration engineers. For linear springs attached to the boundaries, this is a straightforward problem, and the natural frequencies can be calculated using several well-known approaches. One approach that facilitates insight into the way in which the boundaries affect the natural frequencies of the system, is the use of the phase-closure principle. In this paper, this approach is applied to a rod and a beam with nonlinear stiffness boundaries, where the nonlinearity is of the hardening or softening type. The phases of the reflection coefficients, which are a function of frequency and vibration amplitude for a nonlinear boundary, are first calculated. They are then used in the application of the phase closure principle to determine the natural frequencies of the system. To illustrate the approach, examples are presented for both the rod and the beam. It is shown that while the hardening nonlinearity has a marginal effect in both cases, the softening nonlinearity can have a profound effect on the phase of the reflection coefficient, inducing some instabilities.

1. Introduction

The effect of stiffness nonlinearity on the natural frequency of a system is well-known for some systems, most notably lumped parameter systems [1]. A classic example of this type of system, which has a single-degree-of-freedom, is the Duffing oscillator [1,2]. There are several well-known techniques to calculate the natural frequencies of these types of systems, for example the straightforward expansion method, the Lindstedt-Poincaré method, the method of multiple scales, the method of harmonic balance, and methods of averaging [1,3].

Natural frequencies of linear continuous structures, such as rods or beams, and as with lumped parameter systems, can be calculated using well-established analytical methods, e.g. Refs. [4,5]. Some useful tables have been presented for rods and beams in Ref. [6]. More recently, Kang and Kim [7] summarized the work carried out on the vibration characteristics of a flexibly supported beam and plate, and numerically calculated the free vibrations of a thin beam and

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circular plate with general restraints at the boundaries. The Fourier series has also been used to analyse the free vibration of beams with general boundary conditions [8,9]. An alternative approach to this problem is to use the phase closure principle [10–12], (which is also called the "wave-train closure principle" in Ref. [10]). The basic concept is that the total phase change of a travelling wave in a one-dimensional structure during one complete circuit is an integer number times 2π [10–12].

The problem of calculating the natural frequencies of a linear continuous structure with nonlinear stiffness boundaries has not been studied extensively. However, a few research articles on the harmonic, sub-harmonic and super-harmonic responses of the forced response of a continuous beam supported on non-linear springs have been published [13–18]. Dokainish and Kumar [19], and Tabaddor [20] have experimentally investigated the effects of a nonlinear boundary on the lateral vibrations of a beam, and compared the results with theoretical solutions. Ma and da Silva used an iteration method to analyse a beam with nonlinear boundary conditions on elastic foundations [21]. To the authors' knowledge the phase closure principle has not been used to calculate the natural frequencies of such systems.

The aim of this paper is to illustrate the way in which the phase closure principle can be used to calculate the natural frequencies of a linear continuous structure with nonlinear stiffness boundaries. The reason for applying this approach to the nonlinear problem is similar to that for the linear problem, in that shows the way in which the phase change at the boundaries (in this case nonlinear) affects the natural frequencies of the system. Furthermore, it is relatively straightforward to apply. The first step is to calculate the phase change at the boundary, which is done by considering incident and reflected waves at a boundary on a semi-infinite structure. This has been discussed by Vakakis et al. [22,23], for the scattering of longitudinal waves in a rod, and by the authors of this paper for a rod and a beam [24]. Apart from these articles, the literature is devoid of such studies. Once the phase change is known at the boundaries, then this can be combined with the phase closure principle to determine the natural frequencies, which are a function of the amplitude of vibration. In this paper the method is illustrated for both a rod and beam which have a stiffness boundary with softening or hardening nonlinearities.

The paper is organised as follows. Following the introduction, an overview of the problem is presented, and the phase closure principle is outlined. In Section 3, the phase of the reflection coefficient at a boundary is determined for a semi-infinite rod and a beam with nonlinear stiffness supports. The phase closure principle is then used to determine the exact natural frequencies for the finite rod with a nonlinear stiffness boundary, and approximate natural frequencies for the beam with a nonlinear boundary in Section 4. The paper is then concluded in Section 5.

2. Problem statement

Fig. 1 shows a schematic diagram of a rod or a beam of length l with nonlinear boundaries. Natural frequencies of this system occur when the total phase change of a complete circuit of a wave starting from, and returning to, an arbitrary position A, is $2(n+1)\pi,\ \ (n=0,\ 1,\ 2,\ldots)$ [10–12]. The phase change in the beam/rod is $2kl$, in which k is the wavenumber (or the phase change per unit distance). Denoting the phases of the reflection coefficients at the left- and right-hand boundaries as $\phi^{(L)}$ and $\phi^{(R)}$, where the superscripts (L) and (R) indicate the left and right hand boundaries respectively, the condition for a natural frequency is,

$$2kl-\phi^{(L)}-\phi^{(R)}=2(n+1)\pi \quad (n=0,1,2,\ldots). \tag{1}$$

To illustrate this approach, a simple example of a uniform fixed-free rod is considered, in which the left and right nonlinear boundaries in Fig. 1 are fixed and free boundaries respectively. The phase change at the fixed boundary is $-\pi$, and at the free boundary is 0. Under these conditions Eq. (1) becomes $k_L l=(n+1/2)\pi$, $n=0, 1, 2, \ldots$, where $k_L=2\pi f/c_L$ is the longitudinal wavenumber in which c_L is the longitudinal wave speed. This is comparable with the expression $k_L l=(n-1/2)\pi$, $n=1, 2, \ldots$, which is given in Ref. [4]. The natural frequency can then be determined from $f_n=(n+1/2)c_L/2l$, $n=0, 1, 2, \ldots$. More examples about the application of the phase closure principle to finite beams can be found in Ref. [12].

In this paper, nonlinear stiffness boundaries of the hardening and softening type are considered. As, the beam and the rod are linear continuous structures, the phase change in these elements is related to the wavenumber, which is only a function of frequency. However, the phases of the reflection coefficients at the ends of the beam are functions of both amplitude and frequency because of the nonlinear boundaries.

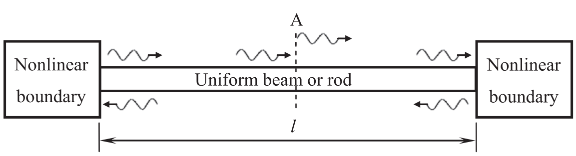

Fig. 1. Finite rod/beam of length l with nonlinear boundaries, showing passage of a wave as it travels a complete circuit starting and finishing at position A.