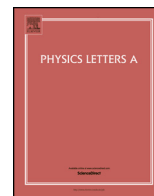




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# Energy content in linear mechanical systems with arbitrary time dependence

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## ABSTRACT

A novel complementary variables formulation, permits the obtainment of invariants of a mechanical system following the rationale of Abel's differential equation identity. This approach allows for the decomposition of the total energy of the system into the energy of the object and the dynamic energy of the field. The force acting on the object must be linear in the spatial variable but is arbitrary in the time variable. Several examples, such as the time dependent harmonic oscillator and a swing are described with this complementary variables formulation. The exact solution for the energy of a Lorentz pendulum with uniformly varying length is presented.

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## 1. Introduction

The time dependent harmonic oscillator (TDHO) has an ubiquitous role in many areas of physics, ranging from particle accelerators dynamics [1] to geodesic equations [2]. Radio frequency ion traps or Paul traps are described by an electric quadrupole potential with sinusoidal time modulation. The Mathieu differential equation describes this time periodic potential and has been extensively studied in Floquet theory. More recently, atomic stabilization in superintense laser fields has led to two forms of atomic stabilization, quasistationary (adiabatic) and dynamic stabilization depending on the laser pulse temporal shape [3]. These schemes ultimately depend on the way the time varying parameter is approximated in the TDHO equation.

The archetypal macroscopic classical mechanical system with a time dependent linear restoring force is the simple pendulum with mass reconfiguration in the small angle approximation [4]. Although similar systems, the pendulum with varying length (self-excited) and a pendulum with varying support (parametrically forced pendulum) exhibit different dynamics even within the perturbation regime [5]. The pumping of a swing has been described using these models, not always leading to consistent results. Energy transfer between the oscillator and the time dependent field is often dealt with qualitative arguments due to the lack of a rigorous formalism.

In the microscopic domain, at the outset of quantum theory, the time dependent harmonic oscillator became all important in order to establish the quantization condition. It was evoked by Henrik Lorentz at the first Solvay conference in 1911, when he posed the problem of a shortening pendulum [6, p. 450]. Two replies were given on that occasion 106 years ago, one by Einstein regarding the adiabatic limit studied by Ehrenfest [7] and the other by Warburg who evoked the abrupt regime undertaken by Galileo. The energy of an atomic ensemble under adiabatic changes was carefully assessed by Ehrenfest and was shown to be consistent with quantum theory [8]. The theory of adiabatic invariants, first developed in statistical and classical mechanics [9,10], was thereafter extended to other areas of physics such as light pulses with slowly varying amplitude envelope. Oddly enough, the name adiabatic was coined in thermodynamics referring to systems without heat transfer. Nonetheless, the way it is used in other fields of physics corresponds to the quasi-static approximation in thermodynamics. A mechanical system with a time dependent force in the opposite limit of abrupt parameter change, was discussed by Galileo while describing 'naturally accelerated motion' [11,12]. The intuitive ideas of energy and constants of motion were introduced at the time, by establishing the constant height which the pendulum bob achieves while varying the length in discrete steps.

The major difficulty to be solved in mechanical systems with explicitly time dependent forces, is to establish the energy transfer between the object and the field. A related problem is finding an invariant proportional to the total energy of the oscillator-field system. The underlying trouble being that it can no longer be

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ascertained that the time dependent Hamiltonian represents the oscillator's energy [13].

In the present paper, a quantity similar to the Wronskian is obtained in section 2 from a second order ordinary differential equation and its derivative. The invariant arising from this complementary variables procedure is related to the total energy of the system. The energy transfer between the oscillator and the time dependent field is obtained from one of the additive terms in the invariant expression. Another conserved quantity using a different pair of complementary variables, derived in section 3, is the Ermakov exact invariant [14,15]. This constant permits the decoupling of the amplitude and phase differential equations as shown in section 4. Several examples are analyzed in section 5 to exhibit the capabilities of the formalism. Subsection 5.4 presents the exact solution to the simple pendulum with uniformly varying length. The interplay between physics and mathematics is decisive in order to establish the appropriate invariants and attribute a physical meaning to the different terms.

2. Energy invariant

Consider the non autonomous second order ordinary differential equation with linear restoring force (LRF)

$$m\ddot{x} + \kappa x = 0, \tag{1}$$

where  $m$  is the object's constant mass,  $x$  describes its position in one spatial dimension, the dots represent differentiation with respect to time and  $-\kappa x$  is the restoring force. The system is required to be linear in the spatial domain. In particular, the force is linear in the spatial coordinate but the stiffness coefficient  $\kappa$  is allowed to have an arbitrary time dependence  $\kappa = \kappa(t)$ . We refer to this equation with the prototypical name as the *time dependent harmonic oscillator* equation. The time dependent potential is  $V = \frac{1}{2}\kappa x^2$ . A differential equation with a first order derivative term  $h\dot{x}$ , can always be eliminated via the transformation  $x \mapsto x_t \exp[-1/2 \int h dt]$ . Therefore, Eq. (1) encompasses the general case. This equation is sometimes written in the literature as

$$\ddot{x} + \Omega^2 x = 0, \tag{2}$$

where  $\Omega^2(t)$  is the time dependent parameter. The dependence of this restoring parameter in terms of other variables depends on the physical system, for example, for the lengthening pendulum  $\Omega^2(t) = \frac{g}{l(t)}$ . Evaluate the derivative of the TDHO equation

$$m\ddot{x} + \kappa\dot{x} + \dot{\kappa}x = 0,$$

that may be written, using the TDHO equation as

$$m\ddot{x} - m\frac{\dot{\kappa}}{\kappa}\ddot{x} + \kappa\dot{x} = 0.$$

Let  $v = \dot{x}$  represent the velocity function, the above equation is then

$$m\ddot{v} - m\frac{\dot{\kappa}}{\kappa}\dot{v} + \kappa v = 0. \tag{3}$$

This is a TDHO type equation but with an extra term involving a first derivative. Although the first derivative could be avoided with the transformation  $v \rightarrow v_t \sqrt{\kappa}$ , it will prove better to retain this term.

The coordinate variable is associated with potential energy whereas the velocity variable is associated with kinetic energy. The position second order ordinary differential equation (ODE) (1) is then related to the evolution of the potential energy of the system, whereas the velocity second order differential equation (3) is

related to the evolution of the system's kinetic energy. Provided that the system is closed, the total energy must be the sum of these two forms of energy. It is in this sense, that the position and velocity variables are complementary. A similar notion between complementary fields has been developed in field theory [16]. Furthermore, we expect the total energy to be conserved, thus, the energy of the system must be the same at all times. Therefore, we seek an appropriate combination of these two equations in such a way that an expression in terms of a total time derivative is obtained. The following procedure renders the required solution.

*2<sup>nd</sup> order ODEs invariant obtention algorithm:*

*Given two second order ordinary differential equations, evaluate the product of the solution of one of them times the other differential equation and viceversa. Take the difference between the two expressions. An invariant is obtained from integration of this result.*

This procedure applied to two solutions of the same second order ODE without first derivative term gives the usual Wronskian expression. If a first order derivative term exists, Abel's differential equation identity is recovered. However, in the present approach, the ODE's are different in general, thus there is no need to consider two linearly independent solutions of the same equation but merely a solution to each of them, possibly the general solution. Invariants obtained by the Sarlet-Bahar method [17,18] can also be derived with this procedure together with the appropriate non-linear transformations.

Evaluate the 2ODEs invariant algorithm beginning with equations (1) and (3), that correspond to the position and velocity variables. For each differential equation, take the product with the solution to the other equation, i.e.  $v$  times Eq. (1)

$$mv\ddot{x} + \kappa vx = 0, \tag{4a}$$

and  $x$  times Eq. (3)

$$mx\ddot{v} - m\frac{\dot{\kappa}}{\kappa}x\dot{v} + \kappa xv = 0. \tag{4b}$$

The difference between these two equations<sup>1</sup> is

$$m\frac{d}{dt}(v\dot{x} - \dot{v}x) + m\frac{\dot{\kappa}}{\kappa}x\dot{v} = 0,$$

where the second derivatives were rewritten using the identity  $v\ddot{x} - x\ddot{v} = \frac{d}{dt}(v\dot{x} - \dot{v}x)$ . This equation is integrated to give the invariant

$$Q_{10} = m(v\dot{x} - \dot{v}x) + \int m\frac{\dot{\kappa}}{\kappa}x\dot{v} dt.$$

The invariant's subindices indicate the order of the derivative in each variable.  $Q_{10}$  implies that  $v$  is a first order derivative of the solution and  $x$  a zeroth order derivative. Since  $\dot{v} = \ddot{x} = -\frac{\kappa}{m}x$ , this invariant can be written as

$$Q_{10} = mv^2 + \kappa x^2 - \int \dot{\kappa} x^2 dt = 2T + 2V - \int \dot{\kappa} x^2 dt. \tag{5}$$

The first two terms are readily recognized to be twice the sum of the kinetic and potential energies of the object. If we abide to the convention for energy with the  $\frac{1}{2}$  factor,

$$\mathcal{E} = \frac{1}{2}Q_{10} = \underbrace{\frac{1}{2}m\dot{x}^2}_{\mathcal{E}_k} + \underbrace{\frac{1}{2}\kappa x^2}_{\mathcal{E}_p} - \underbrace{\frac{1}{2}\int \dot{\kappa} x^2 dt}_{\mathcal{E}_{df}}. \tag{6}$$

<sup>1</sup> Whether the difference (4a)-(4b) is evaluated or the other way around is a matter of convention. We have chosen to subtract (4b) from (4a) to obtain a positive kinetic energy later in the derivation.

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