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Heterogeneity-induced effects for pulse dynamics in FitzHugh–Nagumo-type systems

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HIGHLIGHTS

- Existence of a standing pulse solution with $O(1)$ coefficients.
- Construction of center stable manifolds describing the motion of pulses on heterogeneous media.
- Contraction to a function giving total effects of slowly changing heterogeneity.
- Characterization of the interaction between two pulses.
- Construction of two peak stable stationary solution consisting of two pulse.

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ABSTRACT

Particle like structures have been observed in many fields of science. In a homogeneous medium, a stable, standing pulse is a localized wave that may arise when nonlinear and dissipative effects are in balance. In this paper, we investigate certain phenomena associated with the dynamics of pulse solutions for a FitzHugh–Nagumo reaction–diffusion model. When two pulses are located far from one another initially, their weak interaction drives the subsequent slow dynamics. Our comprehension of the standing pulse profiles allows us to quantitatively characterize their interplay; when the diffusivity of the activator is small compared to that of the inhibitor, the two pulses repel. In addition, using a center-manifold reduction to study the presence of heterogeneities in the environment, we demonstrate that the pulses will move so as to maximize the strength of activation or minimize that of inhibition. The pulse motion will also be influenced by the reaction mechanism.

1. Introduction

Reaction–diffusion equations serve as models for studying various nonlinear phenomena [1–13] in science. One of the interesting features frequently found in reaction–diffusion models is the generation of self-organized patterns. Following on from the pioneering work of Turing [14], pattern formation became an active research field, in which the traditional disciplines of physics, chemistry, biology, and mathematics interact, and significant progress has been made in recent years. Many patterns emerging from homogeneous media are destabilized by a spatial modulation. They consist of basic structure elements like stripes or spots which are more or less uniformly distributed. From a mathematical point of view, when a homogeneous equilibrium loses its stability in tuning a certain physical parameter, the eigenfunction corresponding to the eigenvalue that changes the stability is the dominant mode governing the self-generalized pattern.

Besides these regular patterns found in a neighborhood induced by the Turing instability, localized structures [2,10,15] have also been observed in experiments and numerical simulations. Fronts and pulses are typical examples of localized patterns. A front is a generic structure connecting two different states of a system that possesses a bi-stable nonlinear structure [16–21], whereas a pulse [22–36] is near to a trivial background state superimposed with a number of localized spatial regions where changes are substantial. Such phenomena have been observed, for instance, in the study of nerve pulses in biological systems [5,7,11], concentration drops in

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chemical systems [10], and current filaments [2] in physical systems. We are interested in pulse dynamics related to the FitzHugh–Nagumo equations:

$$u_t = \delta^2 u_{xx} + f(u) - v, \tag{1.1}$$

$$\tau v_t = v_{xx} + u - \gamma v, \tag{1.2}$$

where $\delta, \tau, \gamma \in (0, \infty)$, $f(u) = u(u - \beta)(1 - u)$ and $0 < \beta < 1/2$. As an activator–inhibitor-type reaction–diffusion model, (1.1) and (1.2) received a great deal of attention in studying diffusion-induced instability. Throughout the paper, it is assumed that $\gamma < \frac{3\beta^2}{2(1-2\beta)}$; that is, (1.1) and (1.2) are coupled equations describing a monostable excitable system with $(u, v) = (0, 0)$ being the only constant steady state. A standing pulse of (1.1) and (1.2) is a solution of

$$\delta^2 u_{xx} + f(u) - v = 0, \tag{1.3}$$

$$v_{xx} + u - \gamma v = 0 \tag{1.4}$$

on the real line with asymptotical behavior

$$\lim_{|x|\to\infty} (u(x), v(x)) = (0, 0)\ . \tag{1.5}$$

As both u and v are even functions of x, the existence of standing pulse solutions of (1.1) and (1.2) has been established [22] by variational method. Indeed, for a given $u \in H^1(0, \infty)$, there is a solution of (1.4) denoted by

$$v(x) = \hat{\mathcal{L}}u\ (x) := \int_0^\infty \Gamma(x, s)\, u(s)\, ds, \tag{1.6}$$

where $\Gamma(x, s)$ is the Green function for $-\frac{d^2}{dx^2} + \gamma$ subject to the homogeneous Neumann boundary condition at $x = 0$ and asymptotic decay for large x. Since $\hat{\mathcal{L}} : L^2(0, \infty) \to L^2(0, \infty)$ is self-adjoint, if u_0 is a critical point of $\hat{J}$ defined by

$$\hat{J}(w) := \int_0^\infty \{\frac{\delta^2}{2} w_x^2 + \frac{1}{2} w\, \hat{\mathcal{L}}w - \int_0^{w(x)} f(\xi)\, d\xi\}\ dx\ , \tag{1.7}$$

then $(u_0, \hat{\mathcal{L}}u_0)$ is a standing pulse solution of (1.1) and (1.2). For small δ, as the global minimizer of $\hat{J}$ does not exist, a standing pulse solution was obtained [22] by finding a local minimizer of $\hat{J}$.

Localized standing pulses result from the balance between dissipation and nonlinearity. Such steady states are usually located far from homogeneous equilibrium. The reaction terms of (1.1) and (1.2) are coupled in a skew-gradient structure [37]. Based on this Hamiltonian structure, an index theory [38] was employed to justify the stability [39] of standing pulses of (1.1) and (1.2). For a non-degenerate minimizer u_0 of $\hat{J}$, the stability analysis [39] shows that $(u_0, \hat{\mathcal{L}}u_0)$ is a stable standing pulse of (1.1) and (1.2) if $\tau < \gamma^2$. We refer to [37,39] for some criteria to justify unstable standing pulse and [39–43] for the use of the Maslov index as a tool to study stability of solitary waves. By numerical experiment, the profile of a stable standing pulse of (1.1) and (1.2) is demonstrated in Fig. 1.1.

Heterogeneity is the most important and ubiquitous type of external perturbation observed in natural environments, and indeed natural media generally contain heterogeneity. Nevertheless, in simplified situations, the medium in many mathematical models, on which the process under consideration acts, is assumed to be homogeneous. When a reaction–diffusion system is perturbed by some small heterogeneity, some new phenomena such as the existence of multi-peak solutions [44,45] and the block of traveling pulse solutions [46] have been observed. With a good understanding of standing pulses for the system in a homogeneous medium as stated in the above, we intend to investigate various scenarios of pulse motion in the presence of heterogeneities in the environment.

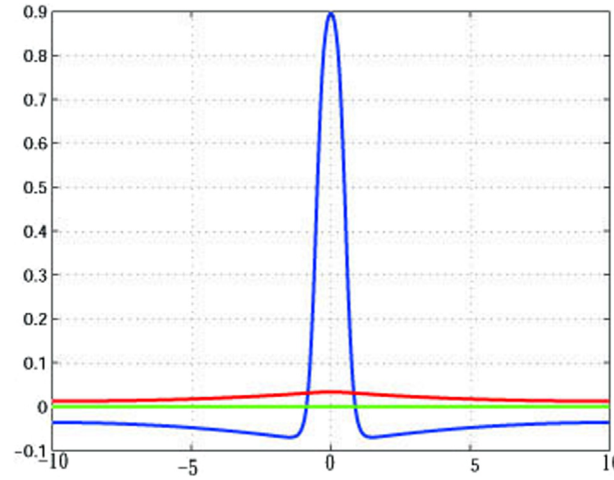

Fig. 1.1. Standing pulse solution u_0 (in blue) changes sign and v_0 (in red) stays positive. Here in (1.3)–(1.4) $\gamma = 0.1$, $\beta = 0.3$ and $\delta = 10^{-2}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In this paper we employ invariant manifold theory to study the following system:

$$\frac{\partial U}{\partial t} = \boldsymbol{F}(U) + \eta \boldsymbol{G}(x), \tag{1.8}$$

where $U := (u, v)$,

$$\boldsymbol{F}(U) := \begin{pmatrix} \delta^2 u_{xx} + f(u) - v \\ \frac{1}{\tau}(v_{xx} + u - \gamma v) \end{pmatrix} \tag{1.9}$$

and η is a small positive number. Instead of dealing with heterogeneous media of jump type [47–49], it is assumed that

$(G1)$ $\boldsymbol{G}(x)$ is a bounded continuous function.

The theory of center-stable manifold has been developed for studying various questions of differential equations (see e.g. [29,50–54] and the references therein). In principle it is effective to deal with a system which has neutral modes, for instance, translation free and rotationally free modes. When a system has a stable family of solutions which are parametrized by free modes, then near such a family of solutions there exist a center-stable manifold in a small perturbed system. This situation frequently appears at a bifurcation point, where the linearization at the trivial solution has critical eigenvalues on the imaginary axis. The parametrized family of solutions is a eigenspace associated with the critical eigenvalues. With certain normal hyperbolicity conditions, the construction of center-stable manifold has been further extended [50,54] to deal with more general family of solutions.

In the study of heterogeneity-induced effects to the motion of pulse dynamics, variational methods are not the main tool for solving the problem; nevertheless it does provide useful information such as the stability and the decay profile of a standing pulse, as to be seen in the next section. As illustrated in Fig. 1.1, we may look at the u component and simply take the peak of a pulse to denote its location, which seems to be convenient to trace the way that each pulse moves. To simplify the analysis in presentation, we assume that

$(G2)$ $\boldsymbol{G} \in (L^2(\boldsymbol{R}))^2$.

The questions to be studied are related to (i) the single pulse motion induced by the heterogeneity (see Theorem 3.1 and (3.9)), (ii) two pulse motion in absence of the heterogeneity (see Theorem 4.1), and (iii) two pulse motion with a heterogeneity (see