Linear Algebra and its Applications 562 (2019) 1–14

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Graded polynomial identities as identities of universal algebras

Yuri Bahturin [a,1], Felipe Yasumura [a,b,*,2]

[a] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C5S7, Canada*
[b] *Department of Mathematics, Universidade Estadual de Campinas, Campinas, SP, 13083-859, Brazil*

A R T I C L E I N F O

Article history:
Received 8 June 2018
Accepted 4 October 2018
Available online 5 October 2018
Submitted by M. Bresar

MSC:
17A42
08B20
16R50

Keywords:
Graded algebra
Graded polynomial identities
Polynomial identities
Universal algebra

A B S T R A C T

Let A and B be finite-dimensional simple algebras with arbitrary signature over an algebraically closed field. Suppose A and B are graded by a semigroup S so that the graded identical relations of A are the same as those of B. Then A is isomorphic to B as an S-graded algebra.

* Corresponding author.
E-mail addresses: bahturin@mun.ca (Y. Bahturin), fyyasumura@mun.ca (F. Yasumura).

[1] Supported by NSERC grant number 227060-14.
[2] Supported by FAPESP grant number 2017/11.018-9.

https://doi.org/10.1016/j.laa.2018.10.006

1. Introduction

In this manuscript, we treat the well-known question whether having the same set of polynomial identities guarantees the isomorphism of algebras. Some obvious restrictions are necessary. In the non-simple case, we have that an algebra A satisfies the same polynomial identities as the sum $A \oplus A$ of two copies of itself. If the field of coefficients is not algebraically closed, then there exist easy examples of non-isomorphic algebras satisfying the same set of ordinary polynomial identities; for example, the algebra of real quaternions $\mathbb{H}$ and the matrix algebra $M_2(\mathbb{R})$ have the same polynomial identities but $\mathbb{H} \not\cong M_2(\mathbb{R})$.

So we need to restrict ourselves to the case of "simple" algebras over algebraically closed fields. Here being simple depends on the full set of structures on the algebras, for example graded-simple, involution-simple, differentially-simple and so on.

In the context of Lie algebras this question was settled by Kushkulei and Razmyslov [21], and in the context of Jordan algebras by Drensky and Racine [12], and Shestakov and Zaicev obtained the result for arbitrary simple algebras [25]. The case of associative algebras is trivial thanks to Amitsur–Levitzki's Theorem.

The case of simple associative algebras graded by abelian group has been resolved by Koshlukov and Zaicev [20]. Having analyzed the structure of G-graded simple associative algebras, G not necessarily abelian, Aljadeff and Haile managed to expand the result of Koshlukov and Zaicev to arbitrary groups [2]. Recently, Bianchi and Diniz studied the problem for arbitrary graded-simple algebras, where the grading is by an abelian group [7].

Imposing various restrictions, this isomorphism question can be investigated in the case of other non-simple algebras. In [11] the authors study the question for certain types of gradings on the associative algebras of upper-block triangular matrices. Also, the classification of group gradings on upper triangular matrices [26] together with the proof of the classification of elementary gradings on the same algebra [9] leads to a positive answer in the context of graded associative algebras of upper triangular matrices.

1.1. *Ω-algebras*

In this paper we deal with so called Ω-algebras, where Ω is a set, called *signature*. One has $\Omega = \bigcup_{m=0}^{\infty} \Omega_m$. An algebra A is called an Ω-algebra, if A is a vector space such that every $\omega \in \Omega_m$ defines an m-ary operation on A, that is, a linear map $\omega : \underbrace{A \otimes \cdots \otimes A}_{m \text{ times}} \to A$. In a natural way, one can define the standard notions of subalgebras, homomorphisms of algebras, ideals, and so on.

Given a non-empty set X, one can define the free Ω-algebra $F = F_\Omega(X)$ as follows. First we build the set $W = W_\Omega(X)$ of Ω-monomials in X as the union of subsets W_n, $n = 0, 1, 2, \ldots$ given by $W_0 = \Omega_0 \cup X$ and for $n > 0$,