Journal of Computational and Applied Mathematics 349 (2019) 1–20

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Anisotropic residual based a posteriori mesh adaptation in 2D: Element based approach

Edward Boey, Yves Bourgault *, Thierry Giordano

Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Avenue, Ottawa, ON, Canada K1N 6N5

ARTICLE INFO

Article history:
Received 10 May 2016
Received in revised form 24 August 2018

Keywords:
A posteriori error estimation
Finite element method
Anisotropic mesh adaptation

ABSTRACT

An element-based adaptation method is developed for an anisotropic a posteriori error estimator. The adaptation does not make use of a metric, but instead equidistributes the error over elements using local mesh modifications. Numerical results are reported, comparing with three popular anisotropic adaptation methods currently in use. It was found that the new method gives favourable results for controlling the energy norm of the error in terms of degrees of freedom at the cost of increased CPU usage. Additionally, we considered a new L^2 variant of the estimator. The estimator is shown to be conditionally equivalent to the exact L^2 error. We provide examples of adapted meshes with the L^2 estimator, and show that it gives greater control of the L^2 error compared with the original estimator.

1. Introduction

In the last twenty years, anisotropic mesh adaptation has seen great activity. Since the work of D'Azevedo and Simpson in [1,2] for piecewise linear approximation of quadratic functions there has been a significant amount of research dedicated to producing practical adaptation procedures based on their results. In addition, there has been much software written for the implementation, which either construct an entirely new mesh, such as BAMG [3], BL2D [4], GAMANIC3D [5], or apply local modifications to a previous mesh, such as MEF++ [6], MMG3D [7], YAMS [8]. The main idea they share in common is to construct a non-Euclidean metric from the Hessian of the solution. We will refer to them as Hessian adaptation methods, see for instance [9–13].

Residual a posteriori error estimation for elliptic equations has been around for some time. In [14,15], Babuska and Rheinboldt introduced a local estimator, constructed entirely from the approximate solution, that is globally equivalent to the energy norm of the error. Numerical results showed that it was suitable for the purposes of mesh adaptation by determining regions in which the mesh could be refined or coarsened. While initially an entirely isotropic method, recently, the residual method was modernized by the introduction of anisotropic interpolation estimates from [16]. Unlike classical results, the new estimates did not require a minimum (or maximum) angle condition, and instead took into account the geometric properties of the element. In [17,18] these interpolation results were combined with the standard a posteriori estimates to drive mesh adaptation by constructing a metric. We will refer to this method as the residual metric method. The method results in highly anisotropic meshes, reducing the error by an order of magnitude compared to isotropic methods [17]. Moreover, the procedure has been successfully applied to a variety of nonlinear situations, including a reaction–diffusion system to model solutal dendrites in [19] and the Euler equations to model the supersonic flow over an aircraft in [20].

* Corresponding author.
E-mail address: ybourg@uottawa.ca (Y. Bourgault).

https://doi.org/10.1016/j.cam.2018.08.053
0377-0427/

Recent work in [21] demonstrates the potential advantages of element-based anisotropic mesh adaptation over the usual metric based mesh adaptation methods used so far. The error estimator they use is hierarchical: from a given approximate solution, they construct a higher-order, more accurate approximation. For the Hessian method it is necessary to take the absolute value of the eigenvalues of the Hessian, thus treating positive and negative curvature as essentially equal, while the distinction can be seen very clearly in meshes adapted with the hierarchical method. Further, the hierarchical estimator has the advantage that it can naturally be applied to finite elements of arbitrary order.

The primary goal of this paper is to introduce, and numerically assess, an element-based adaptation approach to be used with the residual estimator from [17]. We will refer to this method as the element-based residual method. Motivation for implementing such a method includes avoiding the additional steps involved in converting the estimator defined on elements, to a metric defined on the nodes, during which information could be lost. Additionally, we would like to attempt to mimic the success of the hierarchical method. The adaptation will be implemented by interfacing the estimator with the hierarchical adaptation code MEF++. We also introduce a variant of the estimator for the L^2 norm error, which is shown to be reliable and efficient under certain assumptions, and show that the estimator is also suitable for anisotropic mesh adaptation. A secondary goal of the paper will be to provide a comparative performance analysis between four different adaptation techniques: element-based residual, metric based residual, Hessian, and hierarchical.

The outline of this paper is as follows: in Section 2 we introduce the model problem and error estimator, as well as recall some results from the literature; in Section 3 we discuss both the metric and element-based adaptation procedures; in Section 4 we produce numerical results, validating the element-based method, and comparing it with other anisotropic adaptation procedures.

2. The estimator

We discuss the model problem and introduce a residual estimator. Main results will be summarized from the literature. Full details can be found for instance in [16,17,22].

2.1. Model problem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded polygonal domain, with boundary $\partial\Omega$. Let $V = H^1(\Omega)$ and $V_0 = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For $g \in H^{1/2}(\partial\Omega)$, let $V_g = \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\}$, which may be thought of as the translation of V_0 by g. For $f \in L^2(\Omega)$, and a positive definite matrix A, let $u \in V_g$ be the solution of the equation

$$\begin{cases} -\mathrm{div}(A\nabla u) = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Then u is the solution to the variational equation

$$B(u, v) = F(v), \qquad \forall v \in V_0,$$

where

$$B(u, v) = \int_\Omega A\nabla u \cdot \nabla v \, \mathrm{d}x, \qquad u \in V_g,\ v \in V_0,$$
$$F(v) = \int_\Omega f v \, \mathrm{d}x, \qquad v \in V_0.$$

For $h > 0$, let $\mathcal{T}_h$ be a conformal triangulation of Ω consisting of triangles K with diameter $h_K \le h$. Denote by V_h the finite element space of continuous, piecewise linear functions (P_1) on $\mathcal{T}_h$ and $V_{h,0}$ the subspace of functions vanishing on $\partial\Omega$. Let g_h be a piecewise linear approximation of g on $\partial\Omega$ and let $V_{h,g} = \{v_h \in V_h : v_h|_{\partial\Omega} = g_h\}$. Then the finite element approximation $u_h \in V_{h,g}$ of u satisfies the discrete variational equation

$$B(u_h, v_h) = F(v_h), \qquad \forall v_h \in V_{h,0}. \tag{2}$$

For details on the finite element method for elliptic problems, see for instance [23].

2.2. Anisotropic residual error estimator

Define the energy norm by $|||v||| = B(v, v)^{1/2}$ for $v \in V$. The residual mesh adaptation procedure is based on controlling the energy norm of the discretization error $e_h = u - u_h$. The error estimator, which will be outlined below, combines information of the residual with anisotropic interpolation estimates.

Define the localized residual by

$$R_K(u_h) = f + \mathrm{div}(A\nabla u_h),$$