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Packing spanning trees in highly essentially connected graphs

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ABSTRACT

Let $\tau(G)$ be the maximum number of mutually edge-disjoint spanning trees contained in a graph G and let $\kappa'(G)$ denote the edge-connectivity of G. As a corollary of the spanning trees packing theorem by Nash-Williams and Tutte, it is known that if $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$. An edge-cut X of G is an essential edge-cut if $G-X$ contains at least two nontrivial components; and G is essentially k-edge-connected if G does not have an essential edge-cut of size less than k. In this paper, we prove that every g-edge-connected, essentially h-edge-connected graph G with $g \geq k+1$ and $h \geq \frac{g^2}{g-k} - 2$ satisfies $\tau(G) \geq k$. This result is sharp in the sense that there exist infinitely many graphs showing that neither inequality in the hypothesis can be relaxed. Applications to circular flows of graphs, spanning connectivity of line graphs and supereulerian width of graphs are discussed. In particular, we obtained the following, for given integers g and k with $k > 1$ and $2k-1 \geq g \geq k+1$.

(i) Every 5-edge-connected essentially 23-edge-connected graph admits a nowhere-zero 3-flow.
(ii) Every 7-edge-connected essentially 47-edge-connected graph has circular flow number less than 3.
(iii) Every 8-edge-connected essentially 20-edge-connected planar graph has circular 5/2-flow.
(iv) Every g-edge-connected essentially $(\lceil \frac{g^2}{g-k} \rceil - 2)$-edge-connected graph has supereulerian width at least $k+1$.
(v) For a line graph $G = L(H)$, if G is $(\lceil \frac{g^2}{g-k} \rceil - 2)$-connected and $\delta(H) \geq g$, then G is spanning k-connected.

1. Introduction

In this paper, graphs are finite and loopless, but may contain parallel edges. We follow [4] for undefined terminologies and notation. Let $\tau(G)$ be the maximum number of mutually edge-disjoint spanning trees contained in a graph G, and let $\kappa'(G)$ and $\Delta(G)$ denote the edge-connectivity and the maximum degree of G, respectively. For an edge subset $E' \subseteq E(G)$, define the contraction G/E' to be the graph obtained from G by identifying the two end vertices of each edge in E' and then deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. As a widely used application of Nash-Williams and Tutte's theorem [33,37] on spanning tree packing, it is known that

$$\tau(G) \geq \lfloor \frac{\kappa'(G)}{2} \rfloor,$$

as shown by Polesskiĭ [32], Kundu [24] and Catlin [7], among others. We restate it as follows.

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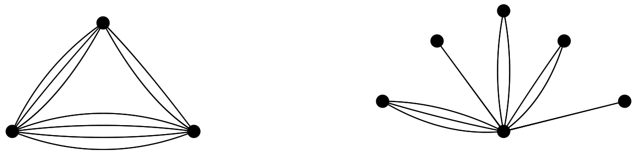

Fig. 1. Some graphs with no essential edge-cut.

Theorem 1.1 (*[33,37]*). *Every* $2k$*-edge-connected graph contains k edge-disjoint spanning trees.*

Examples like K_{2k} indicated that there exist graphs G with $\kappa'(G) = 2k-1$ and $\tau(G) \le k-1$. It is natural to seek conditions on a graph G with $\kappa'(G) < 2k$ which can warrant $\tau(G) \ge k$. The main goal of this paper is to address this problem.

A graph is nontrivial if it contains at least one nonloop edge. An edge-cut X of a connected graph G is **essential** if at least two components of $G - X$ are nontrivial. A graph is **essentially** k**-edge-connected** if it does not have an essential edge-cut with fewer than k edges. It is easy to observe that a loopless connected graph G on $n = |V(G)| \ge 2$ vertices does not have an essential edge-cut if and only if either G is spanned by a K_3 or G has a vertex v_0 such that $E(G - v_0) = \emptyset$ (i.e. the underlying simple graph of G is a star $K_{1,n-1}$), see Fig. 1. For a graph G which is spanned by a K_3, we define the essential edge connectivity of G to be $\Delta(G)$; and if G has a vertex v_0 with $E(G - v_0) = \emptyset$, then define the essential edge connectivity of G to be infinity. As every edge cut of a contraction of G is also an edge cut of G, it follows that the edge connectivity, and the essential edge connectivity are preserved under contraction.

Chartrand and Stewart [8] first introduced the concept of essential edge connectivity as they observed that if $L(G)$ is not complete, then $L(G)$ is k-connected if and only if G is essentially k-edge-connected. In the study of Hamiltonian line graph, Zhan's argument in [42] actually showed that *every* 3*-edge-connected essentially* 7*-edge-connected graph contains two edge-disjoint spanning trees*, which in turn results that every 7-connected line graph is hamiltonian-connected (see [42] or Section 3 for details). The notion of essential edge connectivity is also known as restricted edge connectivity in the literature, as proposed by Esfahanian in [12].

The main result of this paper is the following essential edge connectivity version of Theorem 1.1, which provides a sufficient condition for spanning tree packing.

Theorem 1.2. *Let* k, g, h *be positive integers such that* $k+1 \le g \le 2k-1$ *and* $h \ge \frac{g^2}{g-k} - 2$*. Then every* g*-edge-connected essentially* h*-edge-connected graph contains* k *edge-disjoint spanning trees.*

We remark that the essential edge connectivity condition in Theorem 1.2 is tight as can be seen in Propositions 2.2 and 2.3.

Theorem 1.2 can be applied to circular flows of graphs, and to spanning connectivity of line graphs and to supereulerian width problem of graphs. In the next section, we present the proof of Theorem 1.2. Applications of Theorem 1.2 to circular flows, spanning connectivity of line graphs and to the supereulerian width of graphs will be discussed in Section 3. Our concluding remarks are presented in Section 4.

2. Essential edge connectivity and spanning tree packing

Throughout this section, i and k denote two nonnegative integers. For a graph G, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, $d_i(G) = |D_i(G)|$, $D_{\le k}(G) = \cup_{i \le k} D_i(G)$, and $D_{\ge k}(G) = \cup_{i \ge k} D_i(G)$. When the graph G is understood from the context, we often use D_i, d_i, $D_{\ge k}$ and $D_{\le k}$ for $D_i(G)$, $d_i(G)$, $D_{\ge k}(G)$ and $D_{\le k}(G)$, respectively. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) : u \in U,\ w \in W\}$, and we use $[S, V(G) - S]_G$ to denote an edge-cut of G. We use $E_G(v) = [\{v\}, V(G) - \{v\}]$ to denote a trivial edge-cut for convenience. For any edge $e = uv \in E(G)$, we define $d_G(e) = d_G(u) + d_G(v) - 2$, called the degree of e in G; and $\xi(G) = \min_{e \in E(G)} d_G(e)$, called the **minimum edge degree** of G. The subscript G may be omitted when G is understood from the context. The next theorem will be useful.

Theorem 2.1 (*Li et al. [28] , Xu et al. [39]*). *Every edge-transitive simple graph which is not the star graph has essential edge connectivity equal its minimum edge degree.*

2.1. Tightness of Theorem 1.2

We start with two examples, which would indicate that the conditions $k+1 \le g \le 2k-1$ and $h \ge \frac{g^2}{g-k} - 2$ in Theorem 1.2 are tight.

Proposition 2.2. *For any integer* $k \ge 2$*, and for any sufficiently large integer* $\ell > 0$*, there exists a* k*-edge-connected, essentially* ℓ*-edge-connected graph* G *with* $\tau(G) < k$*.*