Discrete Mathematics 342 (2019) 10–17

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Moore–Penrose inverse of incidence matrix of graphs with complete and cyclic blocks

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ARTICLE INFO

Article history:
Received 30 September 2017
Received in revised form 12 September 2018
Accepted 17 September 2018
Available online xxxx

Keywords:
Moore–Penrose inverse
Block graph
Cut vertex

ABSTRACT

Let Γ be a graph with n vertices, where each edge is given an orientation and let Q be the vertex–edge incidence matrix of Γ. Suppose that Γ has a cut-vertex v and $\Gamma - v = \Gamma[V_1] \cup \Gamma[V_2]$. We obtain a relation between the Moore–Penrose inverse of the incidence matrix of Γ and of the incidence matrices of the induced subgraphs $\Gamma[V_1 \cup \{v\}]$ and $\Gamma[V_2 \cup \{v\}]$. The result is used to give a combinatorial interpretation of the Moore–Penrose inverse of the incidence matrix of a graph whose blocks are either cliques or cycles. Moreover we obtain a description of minors of the Moore–Penrose inverse of the incidence matrix when the rows are indexed by cut-edges. The results generalize corresponding results for trees in the literature.

1. Introduction

For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ the vertex set and the edge set of Γ respectively. Let Γ be a graph with $V(\Gamma) = \{1, 2, \ldots, n\}$, $E(\Gamma) = \{e_1, e_2, \ldots, e_m\}$ and suppose each edge of Γ is assigned an orientation. The vertex–edge incidence matrix of Γ, denoted by $Q(\Gamma)$ (or simply by Q if there is no possibility of a confusion), is the $n \times m$ matrix defined as follows. The rows and the columns of Q are indexed by $V(\Gamma)$, $E(\Gamma)$ respectively. The (i, j)−entry of Q is 0 if vertex i and edge e_j are not incident and otherwise it is 1 or -1 according to as e_j originates or terminates at i respectively.

If A is an $n \times m$ matrix, then an $m \times n$ matrix G is called a generalized inverse of A if $AGA = A$. The Moore–Penrose inverse of A, denoted by A^+, is an $m \times n$ matrix satisfying the equations $AGA = A$, $GAG = G$, $(AG)' = AG$ and $(GA)' = GA$. It is well-known that any complex matrix admits a unique Moore–Penrose inverse. We refer to [4,5] for basic properties of the Moore–Penrose inverse.

The results of [1,3] motivated this work. Recall that a block of a graph Γ is a maximal connected subgraph of Γ that has no cut-vertex [7]. The main purpose of this paper is to obtain a graph-theoretic description of the Moore–Penrose inverse of the incidence matrix of a graph whose blocks are either cliques or cycles. We also obtain a description of minors of the Moore–Penrose inverse in the special case when the rows of the minor are indexed by cut-edges. Since in a tree each block is a complete graph (on 2 vertices) our results generalize the results for trees obtained in [1,3].

Obtaining a graph-theoretic or a combinatorial description of minors of matrices associated with a graph is a topic well-studied in the literature. Indeed the classical matrix-tree theorem and a description of all minors of the Laplacian [6] are examples of such results. Our work is also in a similar spirit.

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https://doi.org/10.1016/j.disc.2018.09.020

In Section 2 we prove some preliminary results. In Section 3 we obtain a general result for the case when Γ has a cut-vertex v. In this case, let $\Gamma - v = \Gamma[V_1] \cup \Gamma[V_2]$. We provide a relation between the Moore–Penrose inverse of the incidence matrix of Γ and the Moore–Penrose inverse of the incidence matrices of the subgraphs induced by $\Gamma[V_1 \cup \{v\}]$ and $\Gamma[V_2 \cup \{v\}]$. Section 4 contains results providing description the Moore–Penrose inverse of the incidence matrix of a graph in which each block is either a complete graph or a cycle. Finally in Section 5, the $k \times k$-minors of the Moore–Penrose inverse of the incidence matrix of an arbitrary graph Γ, whose rows are indexed by cut-edges are obtained.

2. Moore–Penrose inverse

Let $V(\Gamma) = V_1 \sqcup V_2$ be a partition of $V(\Gamma)$ into non-empty disjoint subsets. If the set $\mathcal{K}$ of edges of Γ which have one end in V_1 and one end in V_2 is non-empty, then we say that $\mathcal{K}$ is a edge-cut-set in Γ. We may choose one of the two possible orientations for $\mathcal{K}$, by specifying that one of V_1, V_2 contains the heads of all edges in $\mathcal{K}$, while the other contains the tails. We put $T_{\mathcal{K}}$ and $H_{\mathcal{K}}$, the sets of vertices components of $\Gamma \setminus \mathcal{K}$ which contain the tail and head of $\mathcal{K}$, respectively. Also, we put $t_{\mathcal{K}} = |T_{\mathcal{K}}|$ and $h_{\mathcal{K}} = |H_{\mathcal{K}}|$, where $|.|$ denotes cardinality. The tail characteristic function of $\mathcal{K}$ is defined as

$$\tau_{\mathcal{K}}(v) = \begin{cases} 1, & v \in T_{\mathcal{K}}, \\ 0, & v \in H_{\mathcal{K}}. \end{cases}$$

If $\mathcal{K} = \{e\}$ is a singleton, then we may replace $\mathcal{K}$ by e in $T_{\mathcal{K}}, H_{\mathcal{K}}, t_{\mathcal{K}}, h_{\mathcal{K}}$, and $\tau_{\mathcal{K}}$. Also, the vertex incidence vector $v_{\mathcal{K}}$ of $\mathcal{K}$ is a vector indexed by $V(\Gamma)$ whose vth entry equals 1 if $v \in T_{\mathcal{K}}$, and it is -1 if $v \in H_{\mathcal{K}}$. The edge incidence vector $e_{\mathcal{K}}$ of $\mathcal{K}$ is a vector indexed by $E(\Gamma)$ whose eth entry ($e \in \mathcal{K}$) equals 1 or -1 if its edge-cut-set orientation coincides or reverse with its orientation in Γ, respectively and it is zero elsewhere. We set $\delta_{\mathcal{K}}(e)$ to be the eth entry of $e_{\mathcal{K}}$.

Consider a cycle $\mathcal{C}$ in graph Γ and either clockwise or counterclockwise orientation for $\mathcal{C}$. The edge incidence vector $e_{\mathcal{C}}$ of $\mathcal{C}$ is a vector indexed by $E(\Gamma)$ whose eth entry ($e \in \mathcal{C}$) equals 1 or -1 if its cycle orientation coincides or reverse with its orientation in Γ, respectively and it is zero elsewhere. We set $\delta_{\mathcal{C}}(e)$ to be the eth entry of $e_{\mathcal{C}}$.

The column vector of all ones, of appropriate size, will be denoted by $\mathbf{1}$.

Lemma 2.1. *Let $\mathcal{K}$ be a directed cut in a connected graph Γ and $Q^+(\Gamma) = [q^+_{e,v}]$. Then*

$$\sum_{e \in \mathcal{K}} \delta_{\mathcal{K}}(e) q^+_{e,v} = \tau_{\mathcal{K}}(v) - \frac{t_{\mathcal{K}}}{n}.$$

for all $v \in V(\Gamma)$, where $n = |V(\Gamma)|$.

Proof. Let $e_{\mathcal{K}}$ and $v_{\mathcal{K}}$ be the edge and vertex incidence vectors of the cut $\mathcal{K}$, respectively. Then note that $e'_{\mathcal{K}} = \frac{1}{2} v'_{\mathcal{K}} Q$. It is well-known that QQ^+ is the orthogonal projection on the column space of Q. For a connected graph, the column space of Q equals $\mathbf{1}^\perp$, and hence $QQ^+ = I - \frac{1}{n}J$, where J is the $n \times n$ matrix of all ones, see [2] Lemma 2.15. We obtain

$$e'_{\mathcal{K}} Q^+ = \frac{1}{2} v'_{\mathcal{K}} (I - \frac{1}{n} J) = \frac{1}{2}(v'_{\mathcal{K}} - \frac{1}{n}(t_{\mathcal{K}} - h_{\mathcal{K}})\mathbf{1}') = \frac{1}{2}(v'_{\mathcal{K}} - \frac{1}{n}(2t_{\mathcal{K}} - n)\mathbf{1}'),$$

from which the result follows. □

Since every edge defines an edge-cut-set in a tree, the following result follows immediately.

Corollary 2.2 ([1], Theorem 1)**.** *Let Γ be a tree. Then $Q^+ = [q^+_{e,v}]$, where*

$$q^+_{e,v} = \tau_e(v) - \frac{t_e}{n},$$

for all edges e and vertices v, where $n = |V(\Gamma)|$.

3. Graphs with a cut vertex

Suppose that Γ is a connected graph with a cut-vertex v, and that $\Gamma - v = \Gamma[V_1] \cup \Gamma[V_2]$ be the disjoint union of two subgraphs $\Gamma[V_1]$ and $\Gamma[V_2]$. Let $\Gamma_1 = \Gamma[V_1 \cup \{v\}]$ and $\Gamma_2 = \Gamma[V_2 \cup \{v\}]$, and let Q_1 and Q_2 be the incidence matrices of Γ_1 and Γ_2, respectively. Also, let Q_{1v} (respectively, Q_{2v}) be the reduced incidence matrix obtained by deleting the vth row of Q_1 (respectively, Q_2). Utilizing these notations, the incidence matrix Q of Γ has a block structure as

$$Q = \begin{bmatrix} Q_{1v} & 0 \\ 0 & Q_{2v} \\ -\mathbf{1}'Q_{1v} & -\mathbf{1}'Q_{2v} \end{bmatrix}.$$