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A family of Bell transformations

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ABSTRACT

We introduce a family of sequence transformations, defined via partial Bell polynomials, that may be used for a systematic study of a wide variety of problems in enumerative combinatorics. This family includes some of the transformations listed in the paper by Bernstein & Sloane, now seen as transformations under the umbrella of partial Bell polynomials. Our goal is to describe these transformations from the algebraic and combinatorial points of view. We provide functional equations satisfied by the generating functions, derive inverse relations, and give a convolution formula. While the full range of applications remains unexplored, in this paper we show a glimpse of the versatility of Bell transformations by discussing the enumeration of several combinatorial configurations, including rational Dyck paths, rooted planar maps, and certain classes of permutations.

1. Introduction

Aiming at developing a unifying approach for a variety of enumeration problems, and in the spirit of the work by E. T. Bell on partition polynomials, in this paper we introduce a family of sequence transformations defined via partial Bell polynomials.

Let a, b, c, d be fixed numbers. Given a sequence $x = (x_n)_{n\in\mathbb{N}}$, we let $y = \mathscr{Y}_{a,b,c,d}(x)$ be the sequence defined by

$$y_n = \sum_{k=1}^{n} \frac{1}{n!}\left[\prod_{j=1}^{k-1}(an + bk + cj + d)\right] B_{n,k}(1!x_1, 2!x_2, \dots) \text{ for } n \geq 1, \tag{1}$$

where $B_{n,k}$ denotes the (n,k)-th (exponential) partial Bell polynomial. We call $\mathscr{Y}_{a,b,c,d}(x)$ the Bell transform of x with parameters (a, b, c, d).

For $k = 0, 1, 2, \dots$, the polynomials $B_{n,k}(z_1, \dots, z_{n-k+1})$ may be defined through the series expansion

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} z_j \frac{t^j}{j!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(z_1, z_2, \dots)\frac{t^n}{n!}.$$

These polynomials are homogeneous of degree k, of weight n, and they can be written as

$$B_{n,k}(z_1, \dots, z_{n-k+1}) = \sum_{\alpha\in\pi(n,k)} \frac{n!}{\alpha_1!\alpha_2!\cdots}\left(\frac{z_1}{1!}\right)^{\alpha_1}\left(\frac{z_2}{2!}\right)^{\alpha_2}\cdots,$$

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where $\pi(n,k)$ denotes the set of multi-indices $\alpha \in \mathbb{N}_0^{n-k+1}$ such that

$$\alpha_1 + \alpha_2 + \cdots = k \text{ and } \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n.$$

Note that $B_{n,k}$ contains as many monomials as the number of partitions of $[n] = \{1, \ldots, n\}$ into k parts. Thus, if x enumerates some class of building blocks (with x_j distinct blocks of type j), then the sequence $\mathscr{Y}_{a,b,c,d}(x)$ counts the number of objects that can be made from these building blocks by placing them (according to their type) on a set of partitions induced by the parameters (a, b, c, d). Moreover, the term

$$\frac{1}{n!}\left[\prod_{j=1}^{k-1}(an + bk + cj + d)\right] B_{n,k}(1!x_1, 2!x_2, \ldots)$$

gives the number of such objects made with exactly k blocks. For example, if the induced partitions consist of interval blocks, then the set of resulting objects of length n made with k such blocks is given by

$$\frac{k!}{n!} B_{n,k}(1!x_1, 2!x_2, \ldots). \tag{2}$$

This corresponds to $(a, b, c, d) = (0, 1, -1, 1)$. Moreover, the sum over k from 1 to n gives the known INVERT transform of x, see e.g. [2,4,14], and (2) may be interpreted as the number of colored compositions of n with k parts, where part j comes in x_j different colors.

Another special case of broad interest is the NONCROSSING PARTITION transform, introduced by Beissinger in [2] and systematically studied by Callan in [13]. It corresponds to the parameters $(a, b, c, d) = (1, 0, -1, 1)$, giving $\prod_{j=1}^{k-1}(an+bk+cj+d) = \frac{n!}{(n-k+1)!}$. In this case, (1) becomes

$$y_n = \sum_{k=1}^{n} \frac{1}{(n-k+1)!} B_{n,k}(1!x_1, 2!x_2, \ldots), \tag{3}$$

which counts the configurations obtained by placing the building blocks enumerated by x on top of the noncrossing partitions of $[n]$. In particular, if x is the sequence of ones, denoted by $\mathbb{1} = (1, 1, \ldots)$, then

$$\begin{aligned} y_n &= \sum_{k=1}^{n} \frac{1}{(n-k+1)!} B_{n,k}(1!, 2!, \ldots) \\ &= \sum_{k=1}^{n} \frac{1}{(n-k+1)!} \frac{n!}{k!}\binom{n-1}{k-1} \\ &= \sum_{k=1}^{n} \frac{1}{n}\binom{n}{n-k}\binom{n}{k-1} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Thus $\mathscr{Y}_{1,0,-1,1}(\mathbb{1})$ is the sequence of Catalan numbers that enumerates noncrossing partitions, Dyck paths, rooted trees, and many others combinatorial objects (see e.g. [36]).

The family $\mathscr{Y}_{a,b,c,d}$ also includes several of the known transformations studied by Bernstein and Sloane [4]. For example, EXP, REVERT, and CONV can be realized as instances of BELL.

Our goal is to study $\mathscr{Y}_{a,b,c,d}$ from the algebraic and combinatorial points of view. In Section 2 we give explicit formulas for the inverse $\mathscr{Y}_{a,b,c,d}^{-1}$. In Section 3, we provide equivalent forms of (1) in terms of generating functions. The results in these two sections are obtained using Lagrange inversion together with certain interpolating properties of the partial Bell polynomials proved in [6].

In Section 4 we prove a convolution formula for BELL transforms of the form $\mathscr{Y}_{a,b,c,1}$, and we give a recurrence relation for $\mathscr{Y}_{a,b,-1,1}$ assuming that a and b are not both equal to 0. Section 5 is a compilation of basic examples that showcase special instances of $\mathscr{Y}_{a,b,c,d}$.

In Section 6 we discuss combinatorial applications and give some examples that illustrate how $\mathscr{Y}_{a,b,c,d}$ may be used to link the enumeration of certain classes of combinatorial structures with the enumeration of building blocks that serve as "primitive elements" within each class. In that section, we will focus on rational Dyck paths, rooted planar maps, and certain classes of permutations. For a general approach to the enumeration of irreducible combinatorial objects, we refer to the work by Beissinger [2].

It is not surprising that the results of this paper heavily rely on properties of the partial Bell polynomials. For a list of basic properties and combinatorial identities, we refer the reader to [3,15,18,19,22,29–31,38] and the references therein. In an attempt to make the paper more readable and as self-contained as possible, we have included an appendix with the main technical results on partial Bell polynomials used for our arguments.

2. Inverse relations

We start by proving an interpolating relation for $y = \mathscr{Y}_{a,b,c,d}(x)$, which provides a generalization of Theorem 15 in [6].