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Long paths and toughness of k-trees and chordal planar graphs[☆]

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A B S T R A C T

We show that every k-tree of toughness greater than $\frac{k}{3}$ is Hamilton-connected for $k \geq 3$. (In particular, chordal planar graphs of toughness greater than 1 are Hamilton-connected.) This improves the result of Broersma et al. (2007) and generalizes the result of Böhme et al. (1999).

On the other hand, we present graphs whose longest paths are short. Namely, we construct 1-tough chordal planar graphs and 1-tough planar 3-trees, and we show that the shortness exponent of the class is 0, at most $\log_{30} 22$, respectively. Both improve the bound of Böhme et al. Furthermore, the construction provides k-trees (for $k \geq 4$) of toughness greater than 1.

1. Introduction

We continue the study of Hamiltonicity and toughness of k-trees following Broersma et al. [6] and of chordal planar graphs following Böhme et al. [3].

We recall that for a positive integer k, a *k-tree* is either the graph K_k (that is, the complete graph on k vertices) or a graph containing a vertex whose neighbourhood induces K_k and whose removal gives a k-tree. Clearly, k-trees are chordal graphs. We recall that the *toughness* of a graph G is the minimum, taken over all separating sets X of vertices of G, of the ratio of $|X|$ to the number of components of $G - X$. The toughness of a complete graph is defined as being infinite. We say that a graph is *t-tough* if its toughness is at least t.

In [6], Broersma et al. showed that certain level of toughness implies that a k-tree has a Hamilton cycle (see also [20,26]).

Theorem 1. *Let $k \geq 2$. Every $\frac{k+1}{3}$-tough k-tree (except for K_2) is Hamiltonian.*

In the same paper, they constructed 1-tough k-trees which have no Hamilton cycle for every $k \geq 3$.

An older result considering toughness and Hamiltonicity in another subclass of chordal graphs is due to Böhme et al. [3] who showed the following:

Theorem 2. *Every chordal planar graph (on at least 3 vertices) of toughness greater than 1 is Hamiltonian.*

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In [11], Gerlach generalized Theorem 2 for planar graphs whose separating cycles of length at least four have chords. In this paper, we present a different generalization of Theorem 2 which also improves the result of Theorem 1.

The mentioned results were motivated by the following conjecture stated by Chvátal [9].

Conjecture 3. *There exists t such that every t-tough graph (on at least 3 vertices) is Hamiltonian.*

Conjecture 3 remains open. Partial results are known for some restricted classes of graphs; for instance, for different subclasses of chordal graphs (see [3,5,6,18,19]), and for the class of chordal graphs itself (see [7] or [17]). The best known lower bounds regarding Conjecture 3 for chordal graphs and for general graphs were shown in [2]. The study of toughness of graphs (and Conjecture 3 in particular) is well-documented by a series of survey papers, we refer the reader to [1] (for more recent results, see [4]).

In addition to the result of Theorem 2, Böhme et al. [3] presented 1-tough chordal planar graphs whose longest cycles are relatively short (compared to the number of vertices of the graph); and using the notion of shortness exponent by Grünbaum and Walther [13], they argued the following:

Theorem 4. *The shortness exponent of the class of 1-tough chordal planar graphs is at most $\log_9 8$.*

We recall that the *shortness exponent* of a class of graphs Γ is the lim inf, taken over all infinite sequences G_n of non-isomorphic graphs of Γ (for n going to infinity), of the logarithm of the length of a longest cycle in G_n to base equal to the number of vertices of G_n.

For more results considering the shortness exponent, see the survey [24]. To conclude this section, we mention that by the combination of results of Moser and Moon [22] and Chen and Yu [8], the shortness exponent of the class of 3-connected planar graphs equals $\log_3 2$.

2. New results

We recall that a graph is *Hamilton-connected* if for every pair of its vertices, there is a Hamilton path between them. Clearly, every Hamilton-connected graph (on at least 3 vertices) is Hamiltonian. Using a simple argument, we improve the result of Theorem 1 as follows. (This also improves the result of [20] since Hamilton-connected chordal graphs are, in fact, panconnected.)

Theorem 5. *Let $k \geq 3$. Every k-tree of toughness greater than $\frac{k}{3}$ is Hamilton-connected. Furthermore, every 1-tough 2-tree (except for K_2) is Hamiltonian.*

The proof of Theorem 5 is given in Section 3. We also show that under this toughness restriction a graph is chordal planar if and only if it is a 3-tree or K_1 or K_2 (see Lemma 15). In particular, Theorem 5 implies that chordal planar graphs of toughness greater than 1 are Hamilton-connected (it generalizes the result of Theorem 2).

In the other direction, we present 1-tough chordal planar graphs and 1-tough planar 3-trees whose longest paths and cycles are relatively short.

In particular, for every $\varepsilon > 0$, there exists a 1-tough chordal planar graph G whose longest path has less than $|V(G)|^{\varepsilon}$ vertices. In Section 4, we note that such graphs can be obtained by considering the square of particular trees. Consequently, we adjust the result of Theorem 4 as follows:

Theorem 6. *The shortness exponent of the class of 1-tough chordal planar graphs is 0.*

We remark that the graphs constructed in [3] are 3-connected, so the bound $\log_9 8$ of Theorem 4 also applies to the shortness exponent of the class of 1-tough planar 3-trees (see Lemma 16). In Section 5, we use the standard construction for bounding the shortness exponent (for more details regarding this construction, see for instance [24] or [16]), and we improve this bound by the following:

Theorem 7. *The shortness exponent of the class of 1-tough planar 3-trees is at most $\log_{30} 22$.*

In Section 6, we extend the used construction, and we remark that there are k-trees of toughness greater than 1 whose longest paths are relatively short for every $k \geq 4$. (Meanwhile, 3-trees of toughness greater than 1 are Hamilton-connected by Theorem 5.) This remark slightly improves the lower bound on toughness of non-Hamiltonian k-trees presented in [6], and contradicts the suggestion of [26].

3. Tough enough k-trees are Hamilton-connected

In this section, we prove Theorem 5. Simply spoken, the proof is inductive; we choose a vertex on a path and we extend the path using particular neighbours of this vertex.

For a vertex v, we let $N(v)$ denote its *neighbourhood*, that is, the set of all vertices adjacent to v. We say a set $S \subseteq N(v)$ is a *squeeze* by v if the following properties are satisfied for S and $R = N(v) \setminus S$.