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Note

Infinite family of transmission irregular trees of even order

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ABSTRACT

Distance between two vertices is the number of edges in a shortest path connecting them in a connected graph G. The transmission of a vertex v is the sum of distances from v to all the other vertices of G. If transmissions of all vertices are mutually distinct, then G is a transmission irregular graph. It is known that almost no graphs are transmission irregular. Infinite families of transmission irregular trees of odd order were presented in Alizadeh and Klavžar (2018). The following problem was posed in Alizadeh and Klavžar (2018): do there exist infinite families of transmission irregular trees of even order? In this article, such a family is constructed.

1. Introduction

All graphs considered in this paper are undirected, connected, without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$. By *distance* $d(u, v)$ between vertices $u, v \in V(G)$ we mean the standard distance of a simple graph G, that is, the number of edges on a shortest path connecting these vertices in G. The *transmission*, $tr(v)$, of vertex $v \in V(G)$ is defined as the sum of distances from v to all the other vertices of G. A half of the sum of vertex transmissions gives the *Wiener index* that has found important applications in chemistry (see selected books [6,11–13,16] and reviews [7–10,14,15]). Transmissions of vertices are used for design of distance-based information topological indices [5,6]. The number of different vertex transmissions is known as *Wiener complexity* of a graph [1–3]. A graph is called *transmission irregular* if it has the largest possible Wiener complexity over all graphs of a given order, that is, vertices of the graph have pairwise different transmissions.

It was shown that almost all graphs are not transmission irregular [4]. This follows from the fact that almost every graph has diameter 2. There are infinite families of transmission irregular trees of odd order [4]. The following problem was formulated in [4]: do there exist infinite families of transmission irregular trees of even order? In this paper, we construct such a family.

2. Main result

Consider a tree T_k of order $2k + 6$, $k \geq 3$, depicted in Fig. 1. It contains two growing paths of order k and $k + 1$.

Theorem 1. *If $k = m(m + 1)/2$ where $m = 3$ or $m \geq 8$, then T_k is a transmission irregular tree of even order.*

Tree T_k has order $m(m+1)+6$, $m = 3$ or $m \geq 8$. Initial members of the constructed infinite family are T_6, T_{36}, T_{45} of order 18, 78, 96, respectively. As an illustration, transmissions of trees T_6 and T_{36} are presented. For T_6, we have $tr(v_1), \ldots, tr(v_6) =$

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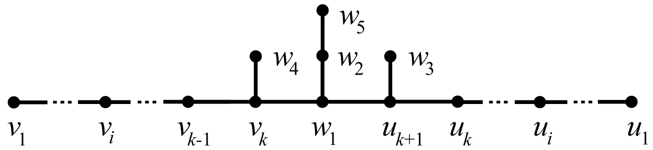

Fig. 1. Tree T_k, $k \geq 3$.

Table 1
Transmissions of T_{36}.

tr	ver	*tr*	ver	*tr*	ver	*tr*	ver	*tr*	ver	*tr*	ver
1376	w_1	1444	u_{31}	1550	v_{26}	1752	u_{20}	2070	v_{13}	2428	u_7
1378	u_{37}	1450	w_2	1554	u_{26}	1788	v_{19}	2074	u_{13}	2490	v_6
1380	v_{36}	1454	w_3	1578	v_{25}	1792	u_{19}	2124	v_{12}	2494	u_6
1384	u_{36}	1456	w_4	1582	u_{25}	1830	v_{18}	2128	u_{12}	2558	v_5
1388	v_{35}	1458	v_{30}	1608	v_{24}	1834	u_{18}	2180	v_{11}	2562	u_5
1392	u_{35}	1462	u_{30}	1612	u_{24}	1874	v_{17}	2184	u_{11}	2628	v_4
1398	v_{34}	1478	v_{29}	1640	v_{23}	1878	u_{17}	2238	v_{10}	2632	u_4
1402	u_{34}	1482	u_{29}	1644	u_{23}	1920	v_{16}	2242	u_{10}	2700	v_3
1410	v_{33}	1500	v_{28}	1674	v_{22}	1924	u_{16}	2298	v_9	2704	u_3
1414	u_{33}	1504	u_{28}	1678	u_{22}	1968	v_{15}	2302	u_9	2774	v_2
1424	v_{32}	1524	v_{27}	1710	v_{21}	1972	u_{15}	2360	v_8	2778	u_2
1428	u_{32}	1526	w_5	1714	u_{21}	2018	v_{14}	2364	u_8	2850	v_1
1440	v_{31}	1528	u_{27}	1748	v_{20}	2022	u_{14}	2424	v_7	2854	u_1

120, 104, 90, 78, 68, 60; $tr(u_1), \ldots, tr(u_7) = 124, 108, 94, 82, 72, 64, 58$; and $tr(w_1), \ldots, tr(w_5) = 56, 70, 74, 76, 86$. Transmissions and the corresponding vertices of T_{36} are collected in Table 1.

3. Proof of Theorem 1

Transmissions of all vertices of T_k will be quadratic polynomials in k. We find explicit form of these polynomials and formulate conditions on k for which two vertices of T_k may have equal transmissions. All other values of k will define an infinite family of transmission irregular trees T_k when k tends to infinity. Let $V = \{v_1, v_2, \ldots, v_k\}$, $U = \{u_1, u_2, \ldots, u_{k+1}\}$, and $W = \{w_1, w_2, \ldots, w_5\}$. Next, we calculate transmissions of vertices of V, U, W and show that no pairs of vertices from these sets have equal transmissions.

1. Calculation of transmissions.

Denote by $t_{i,n}$ transmission of vertex x_i in the simple path of order n with successively numbered vertices $x_1, x_2, \ldots, x_n$. Then $t_{i,n} = i^2 + (n+1)(n-2i)/2$. For vertices $v_i \in V$ and $u_i \in U$, we have

$$tr(v_i) = t_{i,2k+2} + \sum_{j=2}^{5} d(v_i, w_j) = i^2 - (2k+7)i + 2k^2 + 9k + 12,$$

$$tr(u_i) = t_{i,2k+2} + \sum_{j=2}^{5} d(u_i, w_j) = i^2 - (2k+7)i + 2k^2 + 9k + 16.$$

Transmissions of vertices $w_i \in W$ are equal to $tr(w_1) = k^2 + 2k + 8$, $tr(w_2) = k^2 + 4k + 10$, $tr(w_3) = k^2 + 4k + 14$, $tr(w_4) = k^2 + 4k + 16$, and $tr(w_5) = k^2 + 6k + 14$.

2. Comparison of transmissions.

Consider vertices of V and U. If $i \neq j$, we have $i + j - 2k - 7 \leq -6$, and so

$$tr(v_i) - tr(v_j) = tr(u_i) - tr(u_j) = (i-j)(i+j-2k-7) \neq 0$$

and if $i = j$, then

$$tr(v_i) - tr(u_j) = -4 \neq 0.$$

For vertices of W, it is obvious that if $k \geq 3$, then $tr(w_i) \neq tr(w_j)$ for all $i \neq j$. In the next lemma, comparing $tr(w_i)$ with $tr(v_i)$ and $tr(u_i)$, we obtain conditions for T_k to be a non-transmission irregular tree.

Lemma 1. *If there are vertices of T_k with equal transmissions, then*

$$k = m(m+1)/2 - a \ \text{ or } \ k = m(m+1)/4 - b$$

for some integer $m \geq 3$, where $a \in \{3, 5, 6, 7, 8\}$ and $b \in \{5/2, 7/2\}$.