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More complete intersection theorems

Yuval Filmus

Department of Computer Science, Technion — Israel Institute of Technology, Israel

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ABSTRACT

The seminal complete intersection theorem of Ahlswede and Khachatrian gives the maximum cardinality of a k-uniform t-intersecting family on n points, and describes all optimal families. In recent work, we extended this theorem to the weighted setting, giving the maximum μ_p measure of a t-intersecting family on n points. In this work, we prove two new complete intersection theorems. The first gives the supremum μ_p measure of a t-intersecting family on infinitely many points, and the second gives the maximum cardinality of a subset of $\mathbb{Z}_m^n$ in which any two elements x, y have t positions $i_1, \dots, i_t$ such that $x_{i_j} - y_{i_j} \in \{-(s-1), \dots, s-1\}$. In both cases, we determine the extremal families, whenever possible.

1. Introduction

The complete intersection theorem of Ahlswede and Khachatrian [1,3] is a generalization of the classical Erdős–Ko–Rado theorem [10] to the case of t-intersecting families. The theorem states the maximum cardinality of a t-intersecting k-uniform family on n points, for all values of n, k, t. Moreover, it describes all extremal families (in all but a few exceptional cases). The extremal families are of the form $\mathcal{F}_{t,r} = \{S : |S \cap [t+2r]| \geq t+r\}$, where r depends on $\frac{k-t+1}{n}$; the set $[t+2r]$ can be replaced by any set of size $t+2r$.

The complete intersection theorem concerns the setting of k-uniform families. Dinur and Safra [7] considered the weighted setting, in which the aim is to find the maximum μ_p measure of a family on n points without uniformity restrictions, where $\mu_p(A) = p^{|A|}(1-p)^{n-|A|}$. They showed that the original complete intersection theorem implies that when $p < 1/2$, the maximum μ_p measure of a t-intersecting family on an unbounded number of points is $w_{\sup}(t,r) := \max_r \mu_p(\mathcal{F}_{t,r})$. Ahlswede and Khachatrian [2] had considered the case $p = 1/m$ earlier, and their argument (which differs from that of Dinur and Safra) extends to all $p < 1/2$ as well. Recently [11] we have extended these results to all values of p, determining in addition all extremal families; they are all of the form $\mathcal{F}_{t,r}$, and the maximum μ_p measure of a t-intersecting family on n points is $w(n,t,r) := \max_{r \leq \frac{n-t}{2}} \mu_p(\mathcal{F}_{t,r})$.

It is natural to ask what happens when we allow our families to depend on *infinitely* many points rather than on an *unbounded* number of points. In Section 4 we show that when $p < 1/2$, the maximum μ_p measure of a t-intersecting family on infinitely many points is still $\max_r \mu_p(\mathcal{F}_{t,r})$, and furthermore all extremal families are of the form $\mathcal{F}_{t,r}$. We also determine the answer when $p \geq 1/2$.

Theorem 1.1. *Let $t \geq 1$, let $p \in (0,1)$, and let $\mathcal{F}$ be a measurable t-intersecting family on infinitely many points.*

(a) *If $p < 1/2$ then $\mu_p(\mathcal{F}) \leq w_{\sup}(t,p)$. Furthermore, if $\mu_p(\mathcal{F}) = w_{\sup}(t,p)$ then (up to a null set) $\mathcal{F}$ corresponds to an extremal family $\mathcal{F}_{t,r}$.*

E-mail address: yuvalfi@cs.technion.ac.il.

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(b) If $p = 1/2$ then $\mu_p(\mathcal{F}) \le 1/2$. Furthermore, if $\mu_p(\mathcal{F}) = 1/2$ then $t = 1$; in this case $\mathcal{F}$ need not correspond to an extremal family $\mathcal{F}_{t,r}$.

(c) If $p > 1/2$ then $\mu_p(\mathcal{F}) \le 1$, and there is an example of an $\aleph_0$-intersecting family satisfying $\mu_p(\mathcal{F}) = 1$ for all $p > 1/2$.

Ahlswede and Khachatrian [2] considered the analog of their complete intersection theorem to the *Hamming scheme*, in which the objects of study are subsets of $\mathbb{Z}_m^n$ under the uniform measure. Such a subset is *t-agreeing* if any two vectors agree on at least t coordinates. They showed that the original complete intersection theorem implies that the maximum measure of a t-agreeing subset of $\mathbb{Z}_m^n$ for unbounded n is $\max_r \mu_{1/m}(\mathcal{F}_{t,r})$. In Section 5 we extend their work to families in which any two vectors have t coordinates which differ by at most $s - 1$, showing that the maximum measure in this case is $\max_r \mu_{s/m}(\mathcal{F}_{t,r})$. We also determine all extremal families.

Theorem 1.2. *Let $n, m, t \ge 1$ and $s \le m/2$, and let $\mathcal{F}$ be a t-agreeing subset of $\mathbb{Z}_m^n$. The normalized measure of $\mathcal{F}$ is at most $w(n, t, s/m)$. Furthermore, if $s < m/2$ (or $m = 2$, $s = 1$ and $t > 1$) and the normalized measure of $\mathcal{F}$ is exactly $w(n, t, s/m)$, then $\mathcal{F}$ corresponds to an extremal family $\mathcal{F}_{t,r}$.*

The proofs of both results rely on new versions of Katona's circle argument, described in Section 3.

2. Preliminaries

We use $[n]$ for $\{1, \dots, n\}$, $\binom{[n]}{k}$ for all subsets of $[n]$ of size k, and $\binom{[n]}{\ge k}$ for all subsets of $[n]$ of size at least k. We denote by 2^A the set of all subsets of A. The binomial distribution with n trials and success probability p is denoted $\mathrm{Bin}(n, p)$.

We will need the following basic definitions.

Definition 2.1. A *family on n points* is a collection of subsets of $[n]$. A family $\mathcal{F}$ is *t-intersecting* if any two sets in $\mathcal{F}$ have at least t points in common. Two families $\mathcal{F}, \mathcal{G}$ are *cross-t-intersecting* if any set in $\mathcal{F}$ has at least t points in common with every set in $\mathcal{G}$.

A family $\mathcal{F}$ on n points is *monotone* if whenever $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$. Given a family $\mathcal{F}$, its *up-set* $\langle \mathcal{F} \rangle$ is the smallest monotone family containing $\mathcal{F}$, which is $\langle \mathcal{F} \rangle = \{B \supseteq A : A \in \mathcal{F}\}$.

When $t = 1$, we will drop the parameter t: intersecting family, cross-intersecting families.

Definition 2.2. For any $p \in (0, 1)$ and any n, the product measure μ_p is a measure on the set of subsets of $[n]$ given by

$$\mu_p(A) = p^{|A|}(1 - p)^{n-|A|}.$$

For $n \ge t \ge 1$ and $p \in (0, 1)$, the parameter $w(n, t, p)$ is the maximum of $\mu_p(\mathcal{F})$ over all t-intersecting families on n points.

For $t \ge 1$ and $p \in (0, 1)$, the parameter $w_{\sup}(t, p)$ is given by

$$w_{\sup}(t, p) = \sup_n w(n, t, p).$$

It is not hard to see that we can also define $w_{\sup}(t, p)$ as a limit instead of a supremum, since $w(n, t, p)$ is non-decreasing in n. Indeed, every t-intersecting family on n points can be extended to a t-intersecting family on $n + 1$ points having the same μ_p measure.

The optimal families in the weighted complete intersection theorem, named after Frankl [12], are described in the following definition.

Definition 2.3. For $t \ge 1$ and $r \ge 0$, the (t, r)-*Frankl family* on n points is the t-intersecting family

$$\mathcal{F}_{t,r} = \{A \subseteq [n] : |A \cap [t + 2r]| \ge t + r\}.$$

A family $\mathcal{F}$ on n points is *equivalent* to a (t, r)-Frankl family if there exists a set $S \subseteq [n]$ of size $t + 2r$ such that

$$\mathcal{F} = \{A \subseteq [n] : |A \cap S| \ge t + r\}.$$

The following theorem, proved in [11], is the μ_p version of Ahlswede and Khachatrian's complete intersection theorem.

Theorem 2.1. *Let $n \ge t \ge 1$ and $p \in (0, 1)$. If $\mathcal{F}$ is t-intersecting then*

$$\mu_p(\mathcal{F}) \le \max_{r : t+2r \le n} \mu_p(\mathcal{F}_{t,r}).$$

Moreover, unless $t = 1$ and $p \ge 1/2$, equality holds only if $\mathcal{F}$ is equivalent to a Frankl family $\mathcal{F}_{t,r}$.

When $t = 1$ and $p > 1/2$, the same holds if $n + t$ is even, and otherwise $\mathcal{F} = \mathcal{G} \cup \binom{[n]}{\ge \frac{n+t+1}{2}}$ where $\mathcal{G} \subseteq \binom{[n]}{\frac{n+t-1}{2}}$ contains exactly $\binom{n-1}{\frac{n+t-1}{2}}$ sets.