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Analytic continuation as the origin of complex distances in impedance approximations

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ABSTRACT

Engineering approximations of physical systems sometimes produce models in which real-valued model-based physical-distances are added to complex-valued distances and/or (for electrical systems), real-valued current/charge image intensities are replaced with complex-valued quantities. These models are arrived at often using ad hoc approximations that allow infinite integrals or series to be approximated in closed form. Arriving at accurate ad hoc approximations in a compatible analytic form is often the difficult step in the derivation of these approximations. In this paper, we show that this difficult ad hoc step can be replaced for many classes of functions with the use of analytic continuation via Padé approximants, along with some reasonable engineering judgement. We apply our approach to several existing approximations in the electrical engineering field (overhead transmission line impedance, underground cable impedance and Green's functions used in ground potential rise calculations) and show that these approximations can be derived elegantly, without the need for grand leaps of insight, and provide a basis for both distance parameters and current/charge intensities that are complex-valued.

1. Introduction

In engineering derivations, it is not uncommon to encounter infinite series or infinite integrals that have neither a closed form solution nor an analytic representation. While most programming languages handle (closed form) transcendental functions (e.g., trigonometric, logarithmic, exponential, etc.) as well as many non-closed-form analytic functions (gamma function, Bessel function, etc.) we still often run into expressions that must be evaluated by painstaking numerical integration or summation of truncated infinite series (provided we are lucky enough to have convergent series.) Historically, to shorten the execution time of expressions that could not be evaluated strictly with a finite number of elementary operations, it was necessary to find accurate approximations. Even today, when such expressions occur in an iterative process (e.g., the solution of nonlinear equations or optimization) approximations are necessary to make the execution time practical. Finding approximations in an analytic form is often an arduous task that in addition to hard work requires an insightful moment when the researcher hits upon one *compatible* analytic expression that can be substituted for another, allowing the integral to be represented in closed form or an infinite series to be replaced by an expression with a finite number of terms. These approximations are deemed successful when they yield acceptable accuracy over a parameters space that captures the bulk of the practical engineering applications. Developing these approximations historically required intensive trial and error methods [2 4] or numerically fitting the coefficients in the approximations [5] but resulted in surprisingly accurate approximations. What follows is a discussion of several hard-won approximations used for approximating overhead and underground transmission lines, that we will show could have been methodically obtained without relying on epiphanic insight had analytic continuation via Padé approximants been used. Finally, to demonstrate the generality of this approach, we'll apply the technique to the approximation of the Green's functions (functions widely used in physics and engineering problems) encountered when the solution of the electrical ground-grid design/analysis problem is desired for a multilayered earth model.

1.1. Dubanton's insight

Arguably, the most of profound of the insights leading to an accurate approximation across a wide range of frequencies for the conductor impedance with an earth return was that of Dubanton, reported by Gary, giving the approximations in (1) and (2) for the self- and mutual-impedances, respectively, of overhead transmission lines [1]. Note that the integrals in these equations represent the earth correction terms and the definitions of the distance parameters are graphically depicted in

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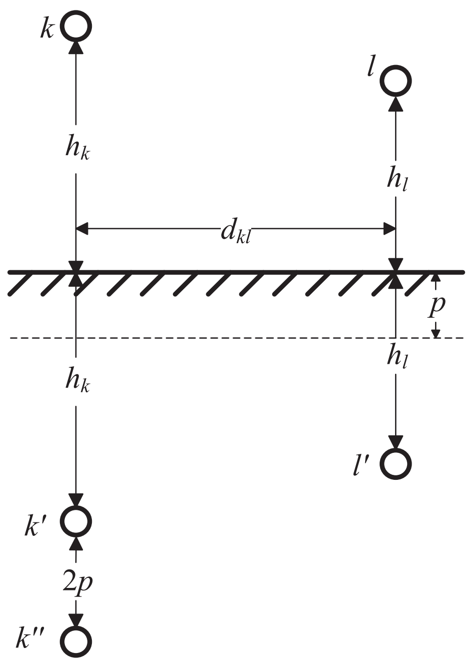

Fig. 1. Overhead line parameter definitions.

Fig. 1, where r is the conductor radius.

$$Z_s = j\omega\frac{\mu_0}{2\pi}\ln\frac{2h}{r} + \omega\frac{\mu}{\pi}\int_0^{\infty}\frac{je^{-2h\lambda}}{\lambda + \sqrt{\lambda^2 + j\omega\mu\sigma}}d\lambda \approx j\omega\frac{\mu_0}{2\pi}\ln\frac{2(h+p_0)}{r}, \quad \mu = \mu_0 \tag{1}$$

$$Z_m = j\omega\frac{\mu_0}{2\pi}\ln\frac{\sqrt{(h_k+h_l)^2 + d_{kl}^2}}{\sqrt{(h_k-h_l)^2 + d_{kl}^2}} + \omega\frac{\mu}{\pi}\int_0^{+\infty}\frac{je^{-(h_k+h_l)\lambda}}{\lambda + \sqrt{\lambda^2 + j\omega\mu\sigma}}\cos\lambda d_{kl}d\lambda \approx j\omega\frac{\mu_0}{2\pi}\ln\frac{\sqrt{(h_k+h_l+2p_0)^2 + d_{kl}^2}}{\sqrt{(h_k-h_l)^2 + d_{kl}^2}}, \quad \mu = \mu_0 \tag{2}$$

The "complex distance" parameter, $p_{(0)}$, is defined by,

$$p = \frac{1}{\sqrt{j\omega\mu\sigma}}, \quad p_0 = \frac{1}{\sqrt{j\omega\mu_0\sigma}} \tag{3}$$

where ω is radian frequency, μ is the earth's permeability, which is assumed to be that of free space, μ_0, and σ is the earth's conductivity.

Gary reports that these approximations were likely obtained by "intuitive insight" and are given without proof. While there is a certain nonchalance about these approximations today, the insight leading to them is quite stunning when viewed from a time that predates them. Dubanton replaced the infinite integral by simply adding a complex-valued distance parameter, in the specific form of p_0, to the height parameter in the "free space" logarithmic term. Why would one predict such an expression would work?

1.2. Deri's et al.'s insight

Deri et al. [2], in a successful attempt to put a derivation behind the Dubanton approximations, invoked the following approximation that also seems to have been intuitively arrived at. (This same approximation—and approach—was used four years later to approximate the integral encountered in derivation of the impedance of underground conductors buried in separate conduits [3].)

$$\frac{2w}{w + \sqrt{1 + w^2}} \approx 1 - e^{-2w} \tag{4}$$

1.3. Alvarado's and Betancourt's insight

Alvarado and Betancourt [4] were able to show that a better approximation than that used by Deri et al., is:

$$\frac{2w}{w + \sqrt{1 + w^2}} \approx 1 - e^{-2w} - \frac{1}{3}w^3e^{-2w} \tag{5}$$

Alvarado remembers that "Yes, we started with Deri. Then we looked at the nature of their error, which looked like a very predictable sort of error…I had been playing with approximation formulas, and when I saw the shape of their error curve we tried to 'fit it' to a formula, but in the end, it was educated wild speculation [7]."

1.4. Wedepohl's insight

Semlyen [8] reported that the work of Deri et al., led L. M. Wedepohl to conjecture that the series self-impedance of a direct-buried underground cable whose insulation thickness was negligible could be approximated by:

$$Z_s \approx j\omega\frac{\mu_0}{2\pi}\ln\frac{r + p_0}{r} \tag{6}$$

1.5. Tylavsky's insight

After becoming familiar with Dubanton's equations and the proof by Deri et al., Tylavsky worked to show that a similar approximation could be obtained for deep underground conductors by first obtaining the theoretically exact expressions for self- and mutual-impedances [9], below, for the geometry shown in Fig. 2, where μ_r is relative permeability of the earth, K_n is a modified Bessel function of the second kind of order n, K'_n, is the derivative of K_n and r is the conductor radius.

$$Z_s = \frac{j\omega\mu_0}{2\pi}\left(\ln\frac{d}{r} + \mu_r\frac{p}{d}\frac{K_0\left(\frac{d}{p}\right)}{K_1\left(\frac{d}{p}\right)} + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{b^2 + br}{d^2}\right)^n \cdot \frac{\mu_r\frac{p}{d}nK_n\left(\frac{d}{p}\right) + K'_n\left(\frac{d}{p}\right)}{\mu_r\frac{p}{d}nK_n\left(\frac{d}{p}\right) - K'_n\left(\frac{d}{p}\right)}\right) \tag{7}$$

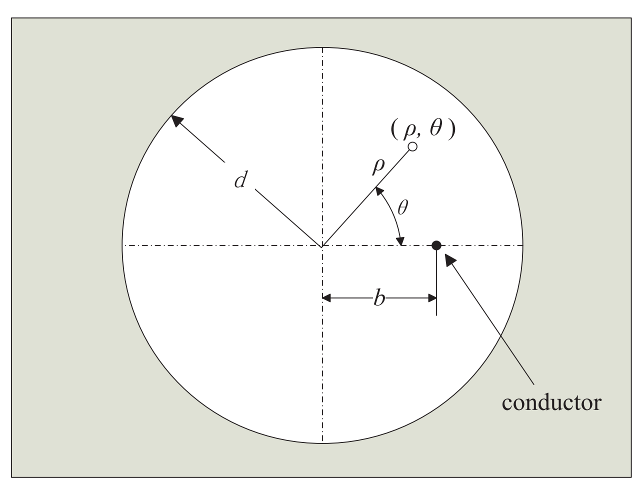

Fig. 2. Underground cable parameter definitions.