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The semi-analytical evaluation for nearly singular integrals in isogeometric elasticity boundary element method

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ABSTRACT

Benefiting from improvement of accuracy in modeling complex geometry and integrity of discretization and simulation, the isogeometric analysis in the boundary element method (IGABEM) has now been implemented by several groups. However, the difficulty of evaluating the nearly singular integral in IGABEM for elasticity has not yet been effectively solved, which will hinder the application of IGABEM in engineering structure analysis. Herein, the nearly singular integrals in IGABEM are separated to the non-singular part and singular part by the subtraction technique. The integral kernels in singular part are approximated by the Taylor series polynomial expressions, in which different orders of derivatives are interpolated by the non-uniform rational B-splines (NURBS). Furthermore, the analytical formulations for the singular part with the approximated kernels are derived by a series of integration by parts, while the non-singular part is calculated with Gaussian quadrature. In this way, a semi-analytical method is proposed for the nearly singular integrals in the IGABEM. Comparing with the conventional IGABEM, the present method can yield accurate displacement and stress for inner points much closer to the boundary. It can obtain effective results with fewer elements than the finite element method because of the precise simulation of geometry and boundary-only discretization.

1. Introduction

It is known that the numerical method, such as the finite element method (FEM), can be very expensive in the discretization when it is used to simulate complex geometries. The isogeometric analysis pioneered by Hughes [1] and further developed by Cottrell [2,3] and Bazilevs [4,5] can overcome this disadvantage since this method bridges computer aided design (CAD) and computer aided engineering (CAE).

The boundary element method (BEM) is another efficient numerical tool, in which only the structure boundary needs to be discretized. The isogeometric analysis has also boosted the development of IGABEM, which can describe complex geometry more accurately than the traditional BEM. The potential problem [6], elasticity problem [7,8], acoustic problem [9] and fracture mechanics problem [10,11] have been studied by using the IGABEM with B-spline elements, NURBS elements and T-spline elements.

In the conventional BEM, it is known that the nearly singular integrals always hinder the accuracy of the numerical results, especially when one calculates the physical quantities of the points which are very close to the actual boundary. That is because for the simulation of the near-boundary points, the Gaussian quadrature used to obtain the integrals consisting of higher order singularities is no longer efficient. To deal with the nearly singular integrals, the researchers have applied different kinds of methods, which can be categorized to the numerical method and analytical method. One of the numerical methods is called the indirect numerical methods which establish new regularized boundary integral equations to avoid calculating the nearly singular integrals [12-15]. For instance, in an object-oriented programing environment INSANE [16], Anacleto et al. [17] presented a self-regular formulation of the elasticity BEM. The other numerical methods are called the direct numerical methods which have been applied to calculate the nearly singular integrals efficiently, such as the interval subdivision method [18,19], special Gaussian quadrature method [20], polynomial transformation method [21,22], distance transformation method, sigmoidal, sinh and exponential transformation methods [23-27]. It is noteworthy that some of the mentioned transformation methods need to employ the distance transformation technique [28] to expand the distance between the source and quadrature points in Taylor series, while this approach is sensitive to the position of the projection point. Therefore, regardless of the projection point location, a new distance transformation technique [29] is implemented. Coupled with the exponential transformation technique, Xie et al. [30] calculated the nearly singular integrals in the elasticity problems. Apart from these numerical methods, the researchers have also deduced the semi-analytical and

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analytical formulations for nearly singular integrals, which can not only obtain accurate results, but also can further reduce the computational cost by using less Gaussian quadrature points. For example, by utilizing piecewise linear test and interpolation functions over flat triangles, Fata [31] developed a semi-analytical treatment for the nearly weakly singular surface integrals in the Galerkin boundary integral equations. By using the integration by parts, Niu, et al. [32,33] proposed a semi-analytical and analytical algorithm for the nearly strong and hyper-singular integrals with linear elements. Recently, Niu et al. [34] have further developed the semi-analytical algorithm of nearly singular integrals in higher-order elements, which makes the simulation for complex geometry more accurate.

All the mentioned techniques above were developed to track the nearly singular integrals in conventional boundary element method. To the best of authors' knowledge, except the exponential transformation [35] and sinh transformation [36] have been respectively applied for the nearly singular integrals of IGABEM in the potential problem and acoustic problem, very few methods are developed to address the nearly singular integrals in an analytical or semi-analytical way. Herein, by approximating the kernels of nearly singular integrals with Taylor series and using a series of integration by parts, a semi-analytical algorithm for elasticity IGABEM is proposed.

2. Non-uniform rational B-splines

The non-uniform rational B-splines are implemented in the IGABEM. First, a short introduction for the B-spline and NURBS is given for completeness.

2.1. Knot vector and basis functions of B-spline

The B-spline basis functions, which are the fundamental of non-uniform rational B-splines, are firstly introduced in this section. Let's define a non-decreasing sequence of parameter values $\Xi=[\xi_1,\xi_2,\cdots,\xi_{n+p+1}](\xi_i\le\xi_{i+1},\ i=1,\ \cdots,\ n+p)$ as a knot vector which denotes the parametric space of the geometry, where ξ_i is known as knot and p denotes the degree. Once the knot vector Ξ is given, the basis functions for the B-spline can be defined in the following recursive way,

$$N_{i,0}(\xi)=\begin{cases}1 & \text{if } \xi_i\le\xi<\xi_{i+1}\\ 0 & \text{otherwise}\end{cases}\quad \text{for } p=0 \tag{1}$$

$$N_{i,p}(\xi)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi)\ \text{for } p=1,2,3,\cdots \tag{2}$$

The basis functions shown in Eqs. (1) and (2) when $p=0$ and $p=1$ are exactly the same with the constant and linear basis functions interpolated in Lagrange elements. Herein, the basis functions with $p\ge2$ are emphasized in the isogeometric analysis. For example, Fig. 1 illustrates the quadratic B-spline basis functions obtained from Eq. (2) with the given knot vector $\Xi=[0, 0, 0, 1, 2, 2, 2]$.

After the basis functions being calculated, the B-spline can be interpolated with all the control points $\boldsymbol{P}_i$ in the following way,

$$\boldsymbol{C}(\xi)=\sum_{i=1}^{n}N_{i,p}(\xi)\boldsymbol{P}_i \tag{3}$$

where n denotes the number of control points.

2.2. Basis functions and derivatives of NURBS

Because of the introduction of weighting for each control point $\boldsymbol{P}_i$, the NURBS can reproduce complex curve and surface more precisely.

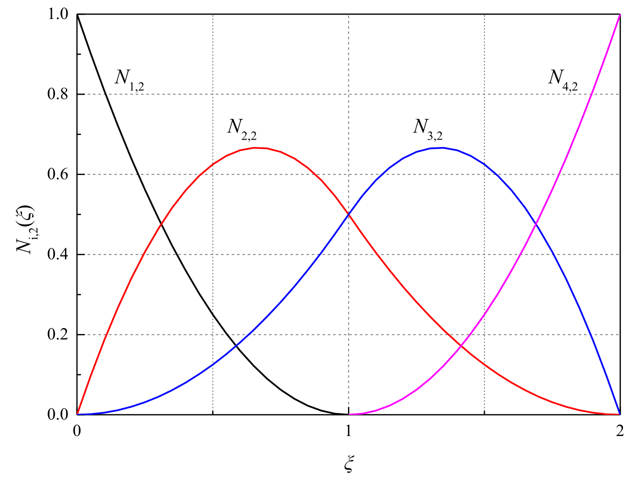

Fig. 1. Quadratic B-spline basis functions with knot vector $\Xi=[0, 0, 0, 1, 2, 2, 2]$.

Similarly to the interpolation of B-spline, the NURBS curve can be interpolated by the following formulation,

$$\boldsymbol{C}(\xi)=\sum_{i=1}^{n}R_{i,p}(\xi)\boldsymbol{P}_i \tag{4}$$

where $R_{i,p}(\xi)$ is the NURBS basis function, which is defined as,

$$R_{i,p}(\xi)=\frac{N_{i,p}(\xi)\omega_i}{\sum_{j=1}^{n}N_{j,p}(\xi)\omega_j} \tag{5}$$

In Eq. (5), $N_{i,p}(\xi)$ and $N_{j,p}(\xi)$ are the B-spline basis functions defined in Eq. (2). It can be concluded that when all the weightings ω are set to unity, the basis functions shown in Eq. (5) will be degenerated to the B-spline basis functions. Therefore, the B-spline can be considered as sub-set of NURBS. The first derivative of NURBS is defined as follows,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}R_{i,p}(\xi)=\omega_i\frac{W(\xi)N'_{i,p}(\xi)-W'(\xi)N_{i,p}(\xi)}{W^2(\xi)} \tag{6}$$

where $N'_{i,p}(\xi)=\mathrm{d}N_{i,p}(\xi)/\mathrm{d}\xi$, $W'(\xi)=\sum_{j=1}^{n}N'_{j,p}(\xi)\omega_j$. And the higher-order derivatives of Eq. (5) can be expressed in terms of lower-order derivatives in a recursive formulation as

$$\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}R_{i,p}(\xi)=\frac{A_i^{(k)}(\xi)-\sum_{b=1}^{k}\binom{k}{b}W^{(b)}(\xi)\frac{\mathrm{d}^{(k-b)}}{\mathrm{d}\xi^{(k-b)}}R_{i,p}(\xi)}{W(\xi)} \tag{7}$$

where the binomial coefficients $\binom{k}{b}$ is defined as $\frac{k!}{b!(k-b)!}$ and $A_i^{(k)}(\xi)=\omega_i\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}N_{i,p}(\xi)$. Let's take a quadratic NURBS, whose knot vector is $\Xi=[0, 0, 0, 1, 2, 2, 2]$ and the weighting is [1, 1, 3, 1], as an example. Its basis functions are illustrated in Fig. 2 and the first two order derivatives are shown in Fig. 3.

3. Isogeometric elasticity boundary element method

The isogeometric analysis for 2D elasticity boundary integral equation in the domain Ω enclosed by the boundary Γ is considered in this section.

The displacements $u_i(\boldsymbol{y})$ and stresses $\sigma_{ik}(\boldsymbol{y})$ at interior point $\boldsymbol{y}$ in the domain Ω can be respectively calculated by the following integral equations,

$$u_i(\boldsymbol{y})=\int_\Gamma\left[U_{ij}^*(x,y)t_j(\boldsymbol{x})-T_{ij}^*(x,y)u_j(\boldsymbol{x})\right]\mathrm{d}\Gamma(\boldsymbol{x})+\int_\Omega U_{ij}^*(x,y)b_j(\boldsymbol{x})\mathrm{d}\Omega \tag{8}$$