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Generalized Noh self-similar solutions of the compressible Euler equations for hydrocode verification

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ABSTRACT

A family of exact self-similar solutions of the compressible Euler equations developed for hydrocode verification is described. This family generalizes the classic Noh problem, which has served as a standard verification test of numerical methods for modeling inviscid compressible flows for three decades. This generalization allows finite pressure initial conditions, nearly arbitrary equations of state, and describes shocked compression as well as isentropic expansion and compression of the gas. In particular, the solutions describe a) the propagation of a finite-strength spherical isentropic expansion wave into a moving uniform gas, leaving behind either a core of uniform gas at rest or a vacuum/cavitation; b) the convergence of a finite-strength isentropic compression wave into a uniform gas or a collapse of a cavity in a finite-pressure gas (a compressible analog of the Rayleigh problem); and c) the expansion of a finite-strength accretion shock wave into a converging isentropic flow of stagnating gas. Our proposed verification test seeks to numerically reproduce all three of these stages of gas motion in a single simulation run. The successful verification of a high-order Godunov Eulerian hydrodynamics code is presented as an example of the expected use of this family of exact solutions.

1. Introduction

The method of exact solutions [1], which compares simulation results with the known solution of an idealized problem, remains a valuable tool of code verification. In one-dimensional compressible gas dynamics, there are a number of problems that permit exact solutions suitable for code verification. The simplest of them is the planar Riemann problem, the evolution of a step-function discontinuity in the initial conditions. For polytropic gases, its solution, first obtained by Kochin [2], is described in detail in [3], §100. (In the literature on code verification, one particularly popular Riemann problem is the Sod shock-tube problem [4,5].) Exact self-similar solutions of Riemann problems involve constant-intensity shock waves and/or centered rarefaction waves and/or a material interface or a tangential discontinuity. For a planar geometry, these solutions provide good verification tests.

For spherical and cylindrical geometries, the simplest verification test is provided by the Noh problem [6]: an initially uniform gas at zero pressure stagnates against a plane, axis, or center of symmetry. (For a planar geometry, the Noh problem obviously reduces to a particular case of a Riemann problem.) Its self-similar solutions contain an expanding accretion shock front that propagates at a constant velocity into a non-uniform flow of a converging zero-pressure gas. These solutions

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have been used for three decades to verify codes designed to model implosion and stagnation. Their main advantage is, as described in [7], "the simplicity of the initial and final conditions in three geometries": all the flow variables have uniform profiles initially, and they remain finite at all times. Moreover, their self-similar profiles are given by explicit analytic formulas. None of the other classic self-similar solutions of the compressible Euler equations, such as Sedov–Taylor–von Neumann blast wave [8,9] or Guderley shock-wave implosion [10], have all these advantages, which makes their use for code verification much more difficult, though not impossible [11,12].

Notwithstanding the apparent simplicity of the Noh problem formulation and numerical initialization, this benchmarking test "poses a formidable challenge to all hydrodynamics codes since they have to deal with an infinitely strong shock" [7], and is able "to dramatically reflect any underlying numerical solver pathologies (e.g., asymmetries and wall-heating phenomena)" [13].

As correctly observed in Ref. [1], the main disadvantage of the method of exact solutions for code verification is that "there are only a limited number of exact solutions available for complex equations (i.e., those with complex geometries, physics, or nonlinearity)." To help address this dearth of exact verification tests, a number of authors have developed new exact solutions of the compressible Euler equations that account for more physics and include complex geometries. In particular, the original Noh solution, henceforth referred to as "classic," has been recently generalized in various ways. A piston/shock tube problem involving multiple shock reflections from the piston and the plane/center/axis of symmetry has been studied in Ref. [7]. The Noh problem for a non-ideal equation of state (EOS), such as Mie–Grüneisen, has been solved in Refs. [13,14]. Generalized solutions of the Noh problem for non-uniform velocity, density and pressure profiles are presented in Ref. [15]. A magnetohydrodynamic (MHD) modification of the Noh problem for a Z-pinch geometry of magnetic field is given in Ref. [16]. A linear perturbation analysis of the classic Noh solutions [17] made it possible to benchmark two- and three-dimensional codes describing the evolution of small perturbations on top of an unsteady stagnation flow in cylindrical and spherical geometry, respectively.

Here, we derive another generalization of the classic Noh solution. We remove the zero-initial-pressure constraint, finding solutions that describe finite-strength shocks. Reproducing such solutions numerically might be less of a "formidable challenge" [7] for code verification than the classic Noh solution, since zero initial pressure can not only be problematic for hydrocodes, but can cause difficulty even for particle-in-cell or kinetic codes when used to model hydrodynamic plasma flows (see, for example, Refs. [18] and [19]), and such codes would also benefit greatly from verification tests. Self-similar solutions of this family include: 1) spherical and cylindrical rarefaction waves analogous to the centered rarefaction waves that can appear in planar Riemann problems; 2) isentropic finite-strength compression waves that steepen as they propagate to eventually form shock waves; 3) both the convergence/implosion phase and the stagnation/expansion phase described by the same solution, similar to the case of the Guderley problem [10,11]. There is also another similarity with the Guderley problem: our new solutions are not given by explicit formulas, they are evaluated using numerical integration of the ordinary differential equations determining the self-similar profiles. As far as code verification is concerned, such semi-analytic solutions are considered "exact" [11] because their numerical accuracy is typically higher than that of the hydrocodes they are used to verify.

Our new solutions are presented for a polytropic ideal gas. They could be further generalized for an ideal fluid with an arbitrary EOS, which does not have to satisfy the Lie symmetry constraints outlined in [9,13,14], and for MHD in a cylindrical Θ-pinch geometry. Both the solutions described here and their possible generalizations permit linear perturbation analysis [17] resulting in explicit analytical dispersion equations. Thus these solutions can be used to generate verification tests for compressible gasdynamic and MHD codes in two and three dimensions, by testing not only the ability of such codes to reproduce the analytic unperturbed one-dimensional solution, but also to reproduce the proper behavior of small-amplitude perturbations applied to that solution in the transverse directions.

This paper is structured as follows. In Section 2, we present a derivation of the new self-similar solutions of the compressible Euler equations. In Section 3, we compare our semi-analytic solutions to the numerical solutions produced by a particular one-dimensional, high-order Godunov Eulerian code, in order to demonstrate how these solutions can be used for hydrocode verification. In Section 4, we conclude with a discussion.

2. Theory

2.1. Formulation of the Noh problem and governing equations

The Noh problem formulation is illustrated with Fig. 1, the terminology and sub-figures of which we refer to extensively in this and the following two paragraphs. Fig. 1(a) schematically shows the initial conditions at $t = 0^-$: uniform gas density ρ_0, pressure p_0, and velocity v_0 directed to the center of symmetry everywhere but at $r = 0$. These initial conditions produce an accretion shock front expanding at a constant velocity v_s at $t > 0$, Figs. 1(b) and (c). The gas behind the shock front is labeled "Uniform shocked gas at rest." The shock front expands into the stagnating incident gas flow whose spherical convergence increases the gas density in the negative radial direction. If the initial gas pressure $p_0 = 0$, then the adiabatic gas compression does not lead to a pressure increase. Hence, the pre-shock gas velocity also stays the same, $-v_0$, as shown in Fig. 1(b), which corresponds to the classic Noh solution. On the other hand, if $p_0 > 0$, then the gas compression leads to an adiabatic pressure increase in the negative radial direction, which, in turn, slows down the converging pre-shock gas, as schematically shown in Fig. 1(c). This situation corresponds to a generalized accretion-shock Noh solution.