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# A discrete-time Pontryagin maximum principle on matrix Lie groups[☆]

Karmvir Singh Phogat *, Debasish Chatterjee, Ravi N. Banavar

*Systems & Control Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India*

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ABSTRACT

In this article we derive a Pontryagin maximum principle (PMP) for discrete-time optimal control problems on matrix Lie groups. The PMP provides first order necessary conditions for optimality; these necessary conditions typically yield two point boundary value problems, and these boundary value problems can then be solved to extract optimal control trajectories. Constrained optimal control problems for mechanical systems, in general, can only be solved numerically, and this motivates the need to derive discrete-time models that are accurate and preserve the non-flat manifold structures of the underlying continuous-time controlled systems. The PMPs for discrete-time systems evolving on Euclidean spaces are not readily applicable to discrete-time models evolving on non-flat manifolds. In this article we bridge this gap and establish a discrete-time PMP on matrix Lie groups. Our discrete-time models are derived via discrete mechanics, (a structure preserving discretization scheme) leading to the preservation of the underlying manifold under the dynamics, thereby resulting in greater numerical accuracy of our technique. This PMP caters to a class of constrained optimal control problems that includes point-wise state and control action constraints, and encompasses a large class of control problems that arise in various field of engineering and the applied sciences.



## 1. Introduction

The Pontryagin maximum principle (PMP) provides first order necessary conditions for a broad class of optimal control problems. These necessary conditions typically lead to two-point boundary value problems that characterize optimal control, and these problems may be solved to arrive at the optimal control functions. This approach is widely applied to solve optimal control problems for controlled dynamical systems that arise in various fields of engineering including robotics, aerospace (Agrachev & Sachkov, 2004; Brockett, 1973; Lee, Leok, & McClamroch, 2008a, b), and quantum mechanics (Bonnard & Sugny, 2012; Khaneja, Brockett, & Glaser, 2001).

Constrained optimal control problems for nonlinear continuous-time systems can, in general, be solved only numerically, and two technical issues inevitably arise. First, the accuracy guaranteed by a numerical technique largely depends on the discretization of the continuous-time system underlying the problem. For control systems evolving on complicated state spaces such as manifolds, preserving the manifold structure of the state space under discretization is a nontrivial matter. For controlled mechanical systems evolving on manifolds, discrete-time models preferably are derived via discrete mechanics since this procedure respects certain system invariants such as momentum, kinetic energy, (unlike other discretization schemes derived from Euler's step) resulting in greater numerical accuracy (Marsden & West, 2001; Ober-Blöbaum, 2008; Ober-Blöbaum, Junge, & Marsden, 2011). Second, classical versions of the PMP are applicable only to optimal control problems in which the dynamics evolve on Euclidean spaces, and do not carry over directly to systems evolving on more complicated manifolds. Of course, the PMP, first established by Pontryagin and his students (Gamkrelidze, 1999; Pontryagin, 1987) for continuous-time controlled systems with smooth data, has, over the years, been greatly generalized, see e.g., Agrachev and Sachkov (2004), Barbero-Liñán and Muñoz Lecanda (2009), Clarke (2013), Clarke (1976), Dubovitskii and Milyutin (1968), Holtzman (1966), Milyutin and Osmolovskii (1998), Mordukhovich (1976), Sussmann (2008) and Warga (1972). However, there is still no PMP that is readily applicable to control systems with discrete-time dynamics evolving on manifolds. As is evident from the preceding discussion, numerical solutions to optimal control problems, via digital computational means, need a discrete-time PMP. Here we establish a PMP for a class of discrete-time controlled systems evolving on matrix Lie groups.

Optimal control problems on Lie groups are of great interest due to their wide applicability across the discipline of engineering: robotics (Bullo & Lynch, 2001), computer vision (Vemulapalli,

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* Corresponding author.
*E-mail addresses:* karmvir.p@iitb.ac.in (K.S. Phogat), dchatter@iitb.ac.in (D. Chatterjee), banavar@iitb.ac.in (R.N. Banavar).

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Arrate, & Chellappa, 2014), quantum dynamical systems (Bonnard & Sugny, 2012; Khaneja et al., 2001), and aerospace systems such as attitude maneuvers of a spacecraft (Kobilarov & Marsden, 2011; Lee et al., 2008b; Saccon, Hauser, & Aguiar, 2013). The conjunction of discrete mechanics and optimal control (DMOC) for solving constrained optimal control problems while preserving the geometric properties of the system has been explored in Ober-Blöbaum (2008). The aforementioned DMOC technique is a direct geometric optimal control technique that differs from our technique on the account that our technique is an indirect method (Trélat, 2012); consequently (Trélat, 2012), the proposed technique is likely to provide more accurate solutions than the DMOC technique. Another important feature of our PMP is that it can characterize abnormal extremals unlike DMOC and other direct methods. Early results on indirect methods for optimal control problems on Lie groups for discrete-time systems derived via discrete mechanics may be found in Kobilarov and Marsden (2011) and Lee et al. (2008a, b) . Another, such technique is to derive higher order variational integrators to solve optimal control problems (Colombo, Ferraro, & Martín de Diego, 2016; Colombo, Jiménez, & Martín de Diego, 2012). It is worth noting that simultaneous state and action constraints have not been considered in any of these formulations. The inclusion of state and action constraints in optimal control problems, while of crucial importance in all real-world problems, makes constrained optimal control problems technically challenging, and, moreover, classical variational analysis techniques are not applicable in deriving first order necessary conditions for such constrained problems (Pontryagin, 1987, p. 3). More precisely, the underlying assumption in calculus of variations that an extremal trajectory admits a neighborhood in the set of admissible trajectories does not necessarily hold for such problems due to the presence of the constraints. This article addresses a class of optimal control problems in which the discrete-time controlled system dynamics evolve on matrix Lie groups, and are subject to simultaneous state and action constraints. We derive first order necessary conditions bypassing techniques involving classical variational analysis. Discrete-time PMPs for various special cases are subsequently derived from the main result.

A discrete-time PMP is fundamentally different from a continuous-time PMP due to intrinsic technical differences between continuous and discrete-time systems (Bourdin & Trélat, 2016, p. 53). While a significant research effort has been devoted to developing and extending the PMP in the continuous-time setting, by far less attention has been given to the discrete-time versions. A few versions of discrete-time PMP can be found in Boltyanskii, Martini, and Soltan (1999), Dubovitskii (1978) and Holtzman (1966).[1] In particular, Boltyanskii developed the theory of tents using the notion of local convexity, and derived general discrete-time PMPs that address a wide class of optimal control problems in Euclidean spaces subject to simultaneous state and action constraints (Boltyanskii, 1975). This discrete-time PMP serves as a guiding principle in the development of our discrete-time PMP on matrix Lie groups even though it is not directly applicable in our problem; see Remark 12 ahead for details.

This article unfolds as follows: our main result, a discrete-time PMP for controlled dynamical systems on matrix Lie groups, and its applications to various special cases are derived in Section 2. Section 3 provides a detailed proof of our main result, and the proofs of the other auxiliary results and corollaries are collected in the Appendices.

[1] Some early attempts in establishing discrete-time PMP in Euclidean spaces have been mathematically incorrect (Bourdin & Trélat, 2016, p. 53).

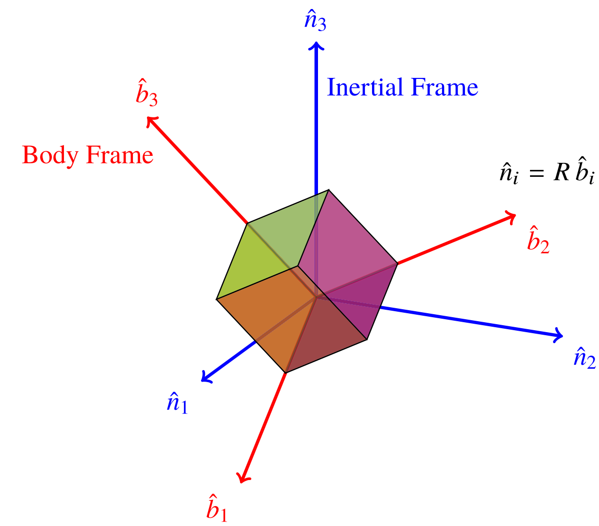


**Fig. 1.** Rigid body orientation.

## 2. Background and main results

This section contains an introduction to Lie group variational integrators that motivates a general form of discrete-time systems on Lie groups. Later in this section we establish a discrete-time PMP for optimal control problems associated with these discrete-time systems.

To illustrate the engineering motivation for our work, and ease understanding, we first consider an aerospace application. Let us first consider an example of control of spacecraft attitude dynamics in continuous time. The configuration space SO(3) (the set of $3 \times 3$ orthonormal matrices with real entries and determinant 1) of a spacecraft performing rotational maneuvers (Lee et al., 2008b), is a matrix Lie group with matrix multiplication as the group operation. Let $R \in SO(3)$ be the rotation matrix that relates coordinates in the spacecraft body frame to the inertial frame, (see Fig. 1,) let $\omega \in \mathbb{R}^3$ be the spacecraft momentum vector in the body frame, and let $u \in \mathbb{R}^3$ be the torque applied to the spacecraft in the body frame. The attitude dynamics in this setting is given in the spacecraft body frame (Lee et al., 2008b) as:

$$\dot{R} = R\hat{\omega}, \tag{1}$$

$$J\dot{\omega} = \hat{\omega}J\omega + u, \tag{2}$$

where $J$ is the $3 \times 3$ moment of inertia matrix of the spacecraft in the body frame, $\hat{\omega} \in \mathfrak{so}(3)$ and $\mathfrak{so}(3)$ (the set of $3 \times 3$ skew-symmetric matrices with real entries) is the Lie algebra (Sachkov, 2009) corresponding to the Lie group SO(3). The first equation (1) describes the kinematic evolution and the second equation (2) describes the dynamics.

Let us, as a first step, uniformly discretize the continuous-time model (1)–(2) to arrive at an approximate discrete-time model. Fixing a step length $h > 0$, we have the discrete-time instances $t \in \{0\} \cup \mathbb{N}$ corresponding to the continuous-time instances $th \in \mathbb{R}$ in a standard way. Therefore, the system configurations at the discrete-time instances defined above are given by

$$R_t := R(th), \quad \omega_t := \omega(th) \quad \text{for all} \quad t \in \{0\} \cup \mathbb{N}.$$

If we assume that spacecraft body momentum is constant on the interval $[th, (t+1)h[$, i.e., $\omega(s) = \omega(th)$ for $s \in [th, (t+1)h[$, then the corresponding kinematic equations $\dot{R}(s) = R(s)\hat{\omega}_t$ for $s \in$