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Brief paper

A kind of LQ non-zero sum differential game of backward stochastic differential equation with asymmetric information ☆

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ABSTRACT

This paper focuses on a kind of LQ non-zero sum differential game driven by backward stochastic differential equation with asymmetric information, which is a natural continuation of Wang and Yu (2010), Wang and Yu (2012). Different from Wang and Yu (2010) and Wang and Yu (2012), a realistic motivation for studying this kind of game is provided, and some feedback Nash equilibrium points are uniquely obtained by forward–backward stochastic differential equations, their filters and the corresponding Riccati equations with Markovian setting.

1. Introduction

Stochastic differential game plays an important role in lots of fields. Many researchers investigated this problem under various setups (Bensoussan, Siu, Yam, & Yang, 2014; Elliott & Siu, 2011; Øksendal & Sulem, 2014; Yong, 2002). Recently, Wang and Yu (2010) studied a non-zero sum differential game of nonlinear backward stochastic differential equation (BSDE). Later, in Wang and Yu (2012), they generalized the game in Wang and Yu (2010) to the partial information case, and obtained an open-loop Nash equilibrium point for an LQ game with same observable information. In some situations of real markets, say, insider trading, one investor may get more information than the others, and then, this investor can make a better decision than the others. It implies that asymmetric information has effect on the decision making. Such a kind of effect is pervasive in reality, but is usually ignored in literature. To fill in the gap, this paper initiates the study of an LQ non-zero sum differential game of BSDE with asymmetric information. This study can be regarded as a first step to investigate such a kind of differential game with asymmetric information.

This paper is closely related to Chang and Xiao (2014) and Shi, Wang, and Xiong (2016), where the state satisfies a (forward) stochastic differential equation (SDE), and thus the BSDE appears as an adjoint of the state equation. In this paper, the state is governed by a BSDE rather than an SDE. Since the construction and property of BSDE are essentially different from those of SDE, the game of BSDE captures different scenarios. See, e.g., Section 2.1 for more information. This paper is also related to Hamadène (1999), Hui and Xiao (2014), Lim and Zhou (2001), Mou and Yong (2006), Yu (2012), Yu and Ji (2008) and Zhang (2011), where asymmetric information is not considered. Therefore, this paper is distinguished from the existing references about stochastic differential game.

The remainder of this paper is organized as follows. In Section 2, a kind of LQ game of BSDE with asymmetric information is formulated in detail and an open-loop Nash equilibrium point is derived. Section 3 is devoted to solving three concrete cases of the LQ game. Feedback Nash equilibrium points are uniquely obtained by the filters of forward–backward SDEs (FBSDEs). To enhance the implementability of the foregoing theoretical results, we work out one numerical example and solve the corresponding numerical solution of the Nash equilibrium point in Section 4. Finally, in Section 5, some concluding remarks are given.

2. Problem formulation and equilibrium points

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space, in which $\mathscr{F}_t$ denotes a natural filtration generated by a two

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dimensional standard Brownian motion $w(t) = (w_1(t), w_2(t))^*$. Suppose that $\mathscr{F} = \mathscr{F}_T$, $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$, and $T > 0$ is a fixed time horizon. We denote by the superscript $*$ the transpose of vectors or matrices, by $|\cdot|$ the norm, and by $\mathscr{F}_t^X$ the filtration generated by a stochastic process X, i.e., $\mathscr{F}_t^X = \sigma\left\{X(s), 0 \le s \le t\right\}$. We call $\mathbb{E}\left(h(t)|\mathscr{F}_t^X\right)$ the optimal filter of $h(t)$ with respect to $\mathscr{F}_t^X$. We also give the notations $\tilde{h}(t) = \mathbb{E}\left(h(t)|\mathscr{F}_t^{w_2}\right)$ and $\hat{h}(t) = \mathbb{E}\left(h(t)|\mathscr{F}_t^{w_1}\right)$.

Let $\mathscr{G}_t^i \subseteq \mathscr{F}_t$ be a given sub-filtration, which represents the information available to the player $i (i = 1, 2)$ up to the time t. If $\mathscr{G}_t^i = \mathscr{F}_t$ (resp. $\mathscr{G}_t^i \subset \mathscr{F}_t$), we call the information available to the player i *complete* (resp. *partial*). If $\mathscr{G}_t^1 \neq \mathscr{G}_t^2$ (resp. $\mathscr{G}_t^1 = \mathscr{G}_t^2$), we call the information available to two players *asymmetric* (resp. *symmetric*). For simplicity, we usually omit the terminology "complete information".

2.1. Economic example

Recursive utility is a generalization of the standard additive utility with an instantaneous utility depending not only on an instantaneous consumption rate, but also on a future utility. Recursive utility problem is one of the hot issues in the current study of economics. In the following, we introduce an optimization problem associated with recursive utility.

Suppose that a consumer has a reward $\xi > 0$ at the terminal time T and continuously consumes between 0 and T. Here ξ is an $\mathscr{F}_T$-measurable and square-integrable random variable. Let $c_1(t)$ and $c_2(t)$ be the consumption rates about two different consumables F_1 and F_2, respectively. Let $p_1(t)$ and $p_2(t)$ be the prices of F_1 and F_2, respectively, which are $\mathscr{F}_t$-adapted processes. Set $\mathscr{P}_t^i = \sigma\{p_i(s); 0 \le s \le t\}$, and let $\mathscr{C}_i = \left\{c_i(\cdot)|\ c_i(t) \text{ is } \mathscr{G}_t^i\text{-adapted and square-integrable}\right\}$ be the set of all consumption rates $c_i(t)$, where $\mathscr{G}_t^i \subseteq \mathscr{P}_t^i$ $(i = 1, 2)$. It implies that the consumer chooses $c_i(t)$ depending on $\mathscr{G}_t^i$ $(i = 1, 2)$. This is reasonable in reality.

Let $y^{c_1,c_2}(t)$ be the Kreps–Porteus recursive utility of the consumer. According to El Karoui, Peng, and Quenez (1997), a special case of $y^{c_1,c_2}(t)$ is modeled by

$$\begin{cases} -dy^{c_1,c_2}(t) = \left(c_1(t) + c_2(t) - y^{c_1,c_2}(t)\right)dt \\ \qquad\qquad\qquad - z_1^{c_1,c_2}(t)dw_1(t) - z_2^{c_1,c_2}(t)dw_2(t), \\ \ \ y^{c_1,c_2}(T) = \xi. \end{cases}$$

Define the performance functional as

$$J_i\big(c_1(\cdot), c_2(\cdot)\big) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \big(c_i(t) - e_i(t)\big)^2 dt + \big(y^{c_1,c_2}(0) - d_i\big)^2\Big],$$

where e_i is a deterministic and uniformly bounded function and d_i is a constant, which are interpreted as two dynamic benchmarks $(i = 1, 2)$. It is natural that the consumer wants to prevent both $c_i(t)$ and $y^{c_1,c_2}(0)$ from large deviation so as to stabilize the recursive utility. That is,

$$\begin{cases} J_1\big(c_1^*(\cdot), c_2^*(\cdot)\big) = \min\limits_{c_1(\cdot)\in\mathscr{C}_1} J_1\big(c_1(\cdot), c_2^*(\cdot)\big), \\ J_2\big(c_1^*(\cdot), c_2^*(\cdot)\big) = \min\limits_{c_2(\cdot)\in\mathscr{C}_2} J_2\big(c_1^*(\cdot), c_2(\cdot)\big). \end{cases}$$

Note that $y^{c_1,c_2}(t)$ satisfies a BSDE and $\mathscr{G}_t^1$ is not always equal to $\mathscr{G}_t^2$. Then the economic example can be regarded as a special LQ non-zero sum differential game of BSDE with asymmetric information.

2.2. Problem formulation

Motivated by the economic example, we consider the controlled linear BSDE

$$\begin{cases} -dy^{v_1,v_2}(t) = \Big[a(t)y^{v_1,v_2}(t) + \sum\limits_{i=1}^2 b_i(t)v_i(t) \\ \qquad + \sum\limits_{i=1}^2 f_i(t)z_i^{v_1,v_2}(t) + c(t)\Big]dt \\ \qquad - z_1^{v_1,v_2}(t)dw_1(t) - z_2^{v_1,v_2}(t)dw_2(t), \\ \ y^{v_1,v_2}(T) = \xi, \end{cases} \tag{1}$$

and the cost functional

$$\mathcal{J}_i\big(v_1(\cdot), v_2(\cdot)\big) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \Big[l_i(t)\big(y^{v_1,v_2}(t) - k_i(t)\big)^2 + m_i(t)\big(v_i(t) - n_i(t)\big)^2\Big]dt + r_i\big(y^{v_1,v_2}(0) - h_i\big)^2\right\} \tag{2}$$

Here a, b_i, f_i, c, k_i and n_i are uniformly bounded and $\{\mathscr{F}_t, 0 \le t \le T\}$-adapted, h_i is a given constant, l_i and m_i are positive, uniformly bounded and $\{\mathscr{F}_t, 0 \le t \le T\}$-adapted, r_i is a nonnegative constant, ξ is an $\mathscr{F}_T$-measurable and square-integrable random variable $(i = 1, 2)$; $v_1(\cdot)$ and $v_2(\cdot)$ are the control processes of the player 1 and the player 2, respectively. Introduce the admissible control set denoted by $\mathscr{U}_i = \left\{v_i(\cdot)|\ v_i(t) \text{ is } \mathscr{G}_t^i\text{-adapted and square-integrable}\right\}$, and each element of $\mathscr{U}_i$ is called an open-loop admissible control for the player $i (i = 1, 2)$. Suppose that each player i hopes to minimize her/his cost functional $\mathcal{J}_i(v_1(\cdot), v_2(\cdot))$ by selecting a suitable admissible control $v_i(\cdot) (i = 1, 2)$. Then the problem is to look for a Nash equilibrium point $(u_1(\cdot), u_2(\cdot)) \in \mathscr{U}_1 \times \mathscr{U}_2$, such that

$$\begin{cases} \mathcal{J}_1(u_1(\cdot), u_2(\cdot)) = \min\limits_{v_1(\cdot)\in\mathscr{U}_1} \mathcal{J}_1(v_1(\cdot), u_2(\cdot)), \\ \mathcal{J}_2(u_1(\cdot), u_2(\cdot)) = \min\limits_{v_2(\cdot)\in\mathscr{U}_2} \mathcal{J}_2(u_1(\cdot), v_2(\cdot)), \end{cases}$$

subject to (1) and (2). We call the game problem an LQ non-zero sum stochastic differential game of BSDE with asymmetric information. For simplicity, we denote the problem by **Problem (AI)**, and abbreviate $(y^{u_1,u_2}, z_1^{u_1,u_2}, z_2^{u_1,u_2})$ by (y, z_1, z_2). Clearly, Problem (AI) covers the example in Section 2.1 as a special case.

The main goal of this paper is to derive some Nash equilibrium points in the feedback form of the filtered states. However, since $\mathscr{G}_t^i$ available to the player $i (i = 1, 2)$ is only an abstract sub-filtration of $\mathscr{F}_t$, it is impossible to obtain feedback Nash equilibrium points in general. Then some special information structures for $\mathscr{G}_t^i (i = 1, 2)$ are desirable to reach the goal. For example, (i) $\mathscr{G}_t^1 = \mathscr{G}_t^2 = \mathscr{F}_t^{w_2}$, i.e., two players have access to the same observation information; (ii) $\mathscr{G}_t^1 = \mathscr{F}_t$ and $\mathscr{G}_t^2 = \mathscr{F}_t^{w_2}$, i.e., one player has more information at any time than the other player; (iii) $\mathscr{G}_t^1 = \mathscr{F}_t^{w_1}$ and $\mathscr{G}_t^2 = \mathscr{F}_t^{w_2}$, i.e., two players have independent observation information and do not share all of their information with each other. These special information structures are inspired by some examples. See Wang, Xiao, and Xiong (2016) for more details.

2.3. Nash equilibrium point

The following proposition is an immediate result of Theorem 2.1 in Wang and Yu (2012). It is very helpful for us to discuss some details and special cases of Problem (AI).

Proposition 2.1. *(u_1, u_2) is a Nash equilibrium point of Problem (AI) if and only if (u_1, u_2) is in the form of*

$$\begin{cases} u_1(t) = \dfrac{\mathbb{E}\left(b_1(t)x_1(t)|\mathscr{G}_t^1\right)}{\mathbb{E}\left(m_1(t)|\mathscr{G}_t^1\right)} + \dfrac{\mathbb{E}\left(m_1(t)n_1(t)|\mathscr{G}_t^1\right)}{\mathbb{E}\left(m_1(t)|\mathscr{G}_t^1\right)}, \\ u_2(t) = \dfrac{\mathbb{E}\left(b_2(t)x_2(t)|\mathscr{G}_t^2\right)}{\mathbb{E}\left(m_2(t)|\mathscr{G}_t^2\right)} + \dfrac{\mathbb{E}\left(m_2(t)n_2(t)|\mathscr{G}_t^2\right)}{\mathbb{E}\left(m_2(t)|\mathscr{G}_t^2\right)}, \end{cases} \tag{3}$$