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Global optimal consensus for discrete-time multi-agent systems with bounded controls☆

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ABSTRACT

In this paper, we consider the global optimal consensus problem for discrete-time multi-agent systems with bounded control protocols over a fixed and directed communication network. Each agent is described by a discrete-time single integrator and endowed with a quadratic objective function which is private to itself. For each agent, we develop two bounded distributed protocols: bounded proportional–integral (PI) protocol and bounded integral (I) protocol, based on the information received from its neighboring agents through the communication network and the gradient of its own objective function. We show that the proposed bounded distributed protocols with properly chosen parameters solve the global optimal consensus problem, i.e., the multi-agent system achieves consensus at a state that minimizes the sum of the objective functions, if the directed communication network is strongly connected and detailed balanced.

1. Introduction

Optimal consensus, which can also be viewed as distributed optimization, where the agents reach a consensus state that minimizes the sum of the objective functions of all agents, has received substantial attention. Various distributed algorithms have been proposed to solve the optimal consensus problem, see, e.g., the survey paper (Nedić, 2015) and the references therein.

However most studies have not considered the physical constraint such as bounded controls. On the other hand, the bound control requirement is realistic in physical applications since every actuator is subject to saturation. Thus, the study of the global optimal consensus problem with bounded controls is of great importance. In Xie and Lin (2017), the authors developed a bounded proportional–integral distributed protocol to solve the global optimal consensus problem for continuous-time single integrators.

Our work is motivated by Xie and Lin (2017) and can be viewed as an extension of its results to the discrete-time setting. More specifically, we consider the global optimal consensus problem, where each agent is described by a discrete-time single integrator and is associated with its own private objective function. For each agent, we develop both a bounded proportional–integral protocol and a bounded integral distributed protocol, based on the gradient of its own objective function and the information received from its neighboring agents through the underlying communication network. We show that these proposed bounded distributed protocols with properly chosen parameters solve the global optimal consensus problem if the directed communication network is strongly connected and detailed balanced.

2. Problem formulation

We consider a multi-agent system of N identical discrete-time single-integrator systems

$$x_i(k+1) = x_i(k) + u_i(k), \quad i \in \mathscr{V} = \{1, 2, \ldots, N\}, \tag{1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and input of agent i, respectively, $|u_i| \leq u_{\max}$ for some positive scalar $u_{\max}$.

Each agent $i \in \mathscr{V}$ has a local convex objective function $f_i : \mathbb{R} \to \mathbb{R}$, which is only known to itself. The global objective function

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of the multi-agent system is $f(x) = \sum_{i=1}^{N} f_i(x)$. The goal of the optimal consensus (distributed optimization) problem is to design distributed protocols, under which, the multi-agent system (1) reaches consensus at a state that minimizes $f(x)$. Throughout the paper, we assume that this optimization problem is feasible. It then follows from Bertsekas (1999) that, if the local objective functions are strictly convex, then there exists a unique optimizer $x^* \in \mathbb{R}$. Moreover, the necessary and sufficient optimality condition is $\nabla f(x^*) = \sum_{i=1}^{N} \nabla f_i(x^*) = 0$.

The local communication is described by a directed weighted graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, with the set of agents $\mathscr{V} = \{1, \dots, N\}$, the set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and the weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise. We also assume that there are no self-loops, i.e., $a_{ii} = 0$ for $i \in \mathscr{V}$.

We make the following assumptions.

Assumption 1. The digraph $\mathscr{G}$ is strongly connected and detailed balanced.

For a digraph satisfying Assumption 1, there exist some real numbers $\omega_i > 0, i = 1, 2, \dots, N$, such that the coupling weights of the graph satisfy $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j = 1, 2, \dots, N$ (Jiang & Wang, 2009). We next define $\tilde{L} = \text{diag}\{\omega\}L$, where $\text{diag}\{\omega\} = \text{diag}\{\omega_1, \omega_2, \dots, \omega_N\}$, and L is the Laplacian matrix associated with the directed graph, defined as $L = [\ell]_{ij} \in \mathbb{R}^{N\times N}$ with $\ell_{ii} = \sum_{j=1}^{N} a_{ij}$ and $\ell_{ij} = -a_{ij}$ for $j \neq i$. Note that $\tilde{L}$ is a symmetric Laplacian matrix.

Assumption 2. The objective function $f_i : \mathbb{R} \to \mathbb{R}, i \in \mathscr{V}$, is quadratic and is given by $f_i(x) = a_i x^2 + b_i x + c_i$, where $a_i > 0$.

We aim to solve the global optimal consensus problem for the multi-agent system (1) with bounded protocols. More specifically, for each agent $i \in \mathscr{V}$, we design a bounded distributed protocol u_i with $|u_i| \le u_{\max}$, under which, the multi-agent system achieves consensus at a state x^*, i.e., $\lim_{k\to\infty} x_i(k) = x^*$ for all $i \in \mathscr{V}$.

3. Main results

Inspired by the bounded distributed continuous-time protocol proposed in Xie and Lin (2017), we design the following bounded distributed control protocol,

$$v_i(k+1) = v_i(k) + \alpha\beta\omega_i \sum_{j=1}^{N} a_{ij}(x_i(k) - x_j(k)),\ v_i(0) = 0, \tag{2a}$$

$$u_i(k) = \sigma_\Delta\Bigg(-v_i(k) - \alpha\nabla f_i(x_i(k)) - \beta\omega_i \sum_{j=1}^{N} a_{ij}(x_i(k) - x_j(k))\Bigg),\ i \in \mathscr{V}, \tag{2b}$$

where $\alpha, \beta > 0$ are gain parameters, and for a given positive scalar $\Delta > 0$, $\sigma_\Delta : \mathbb{R} \to \mathbb{R}$ is a saturation function with a saturation level Δ, i.e., for $s \in \mathbb{R}$, $\sigma_\Delta(s) = \text{sgn}(s)\min\{\Delta, |s|\}$. Note that we can set the value of $\Delta = u_{\max}$ so that the control input is bounded by $|u_i| \le u_{\max}$.

Next we present our main results whose proofs are given in the appendix.

Lemma 1. *Consider the multi-agent system (1). Assume that Assumptions 1 and 2 are satisfied. Under the bounded distributed protocol (2), the equilibrium point of the closed-loop system minimizes the global objective function $f(x) = \sum_{i=1}^{N} f_i(x)$.*

In order to present the next result, we define $A = \text{diag}\{a_1, a_2, \dots, a_N\}$, where $a_i > 0$ is the coefficient for the quadratic term of the objective function $f_i(\cdot)$ for agent i, and the matrix

$$M = \left(\beta(\alpha - 1)\tilde{L} + (I_N - 2\alpha A)\right)^2 + \alpha\beta\tilde{L} - I_N. \tag{3}$$

Theorem 1. *Consider the multi-agent system (1). Assume that Assumptions 1 and 2 are satisfied. Then the bounded distributed protocol (2) with α and β satisfying $\alpha\beta < \frac{1}{\lambda_{\max}(\tilde{L})}$ and $M < 0$, where $\lambda_{\max}(\tilde{L})$ is the largest eigenvalue of the symmetric Laplacian matrix $\tilde{L}$, solves the global optimal consensus problem, i.e., $\lim_{k\to\infty} x_i(k) = x^*$ for all $i \in \mathscr{V}$.*

Remark 1. Note that the linear matrix inequality (LMI) $M < 0$ is always solvable by properly choosing the gain parameters α and β.

Note that the proposed bounded distributed protocol (2) is a proportional–integral controller. Let us now consider the following simplified bounded distributed protocol involving only the integral control action,

$$v_i(k+1) = v_i(k) + \alpha\beta\omega_i \sum_{j=1}^{N} a_{ij}(x_i(k) - x_j(k)),\ v_i(0) = 0, \tag{4a}$$

$$u_i(k) = \sigma_\Delta\Big(-v_i(k) - \alpha\nabla f_i(x_i(k))\Big). \tag{4b}$$

We have the following result whose proof is omitted since it is similar to that of Theorem 1.

Theorem 2. *Consider the multi-agent system (1). Assume that Assumptions 1 and 2 are satisfied. Then the bounded distributed protocol (4) with α and β satisfying $\alpha\beta < \frac{1}{\lambda_{\max}(\tilde{L})}$ and $M_I < 0$, where $M_I = \left(\beta\alpha\tilde{L} + (I_N - 2\alpha A)\right)^2 + \alpha\beta\tilde{L} - I_N$, solves the global optimal consensus problem.*

4. Conclusions

This paper considered the global optimal consensus problem with bounded controls for a multi-agent system, where each agent is a discrete-time single integrator and has a quadratic cost function that is only known to itself. We developed two bounded distributed protocols and showed that the proposed distributed protocols with properly chosen parameters solve the global optimal consensus problem if the directed communication network is strongly connected and detailed balanced.

Appendix A. Proof of Lemma 1

Let $\mathbf{x}(k) = [x_1(k), \dots, x_N(k)]^{\mathrm{T}}$, $\mathbf{v}(k) = [v_1(k), \dots, v_N(k)]^{\mathrm{T}}$, and $F(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}$, $F(\mathbf{x}) = \sum_{i=1}^{N} f_i(x_i)$. From (1) and (2), we obtain the following dynamics in a compact form,

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \sigma_\Delta\left(-\mathbf{v}(k) - \alpha\nabla F(\mathbf{x}(k)) - \beta\tilde{L}\mathbf{x}(k)\right),$$
$$\mathbf{v}(k+1) = \mathbf{v}(k) + \alpha\beta\tilde{L}\mathbf{x}(k),\ \mathbf{v}(0) = \mathbf{0}, \tag{A.1a}$$

where we have abused the notation by using σ_Δ to also denote a vector valued saturation function, i.e., for $\mathbf{s} = [s_1, s_2, \dots, s_N]^{\mathrm{T}} \in \mathbb{R}^N$, $\sigma_\Delta(\mathbf{s}) = \left[\sigma_\Delta(s_1), \sigma_\Delta(s_2), \dots, \sigma_\Delta(s_N)\right]^{\mathrm{T}} \in \mathbb{R}^N$ and $\nabla F(\mathbf{x}) = \left[\nabla f_1(x_1), \nabla f_2(x_2), \dots, \nabla f_N(x_N)\right]^{\mathrm{T}}$.

Note that $\mathbf{1}^{\mathrm{T}}\tilde{L} = 0$. Therefore, left multiplying (A.1a) by $\mathbf{1}^{\mathrm{T}}$ gives $\mathbf{1}^{\mathrm{T}}\mathbf{v}(k+1) = \mathbf{1}^{\mathrm{T}}\mathbf{v}(k) + \alpha\beta\mathbf{1}^{\mathrm{T}}\tilde{L}\mathbf{x}(k) = \mathbf{1}^{\mathrm{T}}\mathbf{v}(k)$. This implies that the term $\sum_{i=1}^{N} v_i(k)$ remains unchanged with respect to time. Thus,

$$\sum_{i}^{N} v_i(k) = \sum_{i=1}^{N} v_i(0) = 0,\ \forall k = 0, 1, \dots \tag{A.2}$$

Let us denote the equilibrium point of the closed-loop multi-agent system (A.1) as $(\mathbf{x}^e, \mathbf{v}^e)$, where $\mathbf{x}^e = [x_1^e, x_2^e, \dots, x_N^e]^{\mathrm{T}}$ and